Coordination Frictions and Heterogeneity in Markets with Bidding

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Abstract

Two heterogeneous buyers with commonly known preferences must choose which one of two different goods–a high value good and a low value good–to bid on when the goods are sold through simultaneously held first-price auctions. We find that efficiency of equilibrium allocation depends on allowing sellers to announce and commit to reservation prices before the auctions are held. This is achieved by first characterizing a unique buyer equilibrium for a generic subset of the parameter space when reservation prices are exogenously set to zero. This equilibrium exhibits coordination frictions and results in a potentially inefficient allocation of the goods. The high value good is sold for sure, though it could be to either buyer and could even be at the reservation price. The low value good is either not sold or sold exactly at the reservation price to the buyer with a lower marginal valuation for the high value good. We then characterize buyer equilibria for all reservation prices. Endogenizing seller behavior by allowing sellers to announce and commit to reservation prices results in a unique market equilibrium of the game in which buyers perfectly coordinate, the efficient allocation is achieved, and both allocation and payoffs are equivalent to that of the price-posting case. The contrasting results between exogenous and endogenous reservation prices stems from the following: when the same side that searches (here, the buyer side) is responsible for price formation, coordination frictions are still present because buyers have incentive to compete over the higher value good and hence could both bid on it. However, when the passive side of the market (here, the seller side) is in effect responsible (here, by setting binding reservation prices), the two-sided heterogeneity results in perfect coordination by the buyers in choosing their goods.

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1 Introduction

This paper considers a market in which two heterogenous buyers who wish to consume one good are constrained to choose between two first-price auctions selling different goods, one of which the buyers agree is of higher value. Two key features of the market structure considered here is that (1) sellers are exogenously forced to sell their goods simultaneously to capacity constrained buyers, and (2) buyers choose which goods to apply to and then bid to determine the final price without ever learning which good the other buyer is bidding on.

The first feature is motivated by several real world markets. Consider for example the market for houses. A selling homeowner may be better off under certain conditions if all potential buyers were able to pursue her house before some decided to move on to the next one. In reality, there are many other homes being sold at exactly the same time, and coordination frictions inevitably arise. These frictions result in multiple buyers attempting to purchase one house while another house is left uncontested. The academic job market is another example in which many sellers (workers who wish to sell their labor) are forced to sell simultaneously to capacity constrained buyers (universities)[1]. Such a mechanism generally leads to inefficient allocation due to the same type of coordination frictions.

The second feature of the market, that buyers–and not sellers–determine the equilibrium prices, is in contrast to a vast majority of the directed search literature. This literature–Burdett, Shi, and Wright (2001) and Shimer (2005) are two of many examples–has focused primarily on cases in which one side of the market posts locations and prices, while the other side observes these prices and decides where to apply. In goods market applications, it is typically the sellers who post prices and consumers who search. In directed labor search, firms post wages and workers search. This paper focuses on cases in which the *same* side of the market–the buyer side–is in effect responsible for both searching and determining the equilibrium price. To our knowledge, the work closest to considering such a mechanism is Julien, Kennes, and King (2000). However, their (labor market) model, firms decide in a preliminary stage which workers to "bid" on, and then observe how many other firms are bidding on the same worker before choosing their bids.[2] Our assumption that buyers must choose their bids without learning if they have competition makes this model differ greatly from theirs. Instead, equilibrium bidding behavior in this framework features buyers choosing their bids from continuous distributions with identical and connected support. Technically, this result is similar to that of Burdett and Judd (1983), in which firms post prices without knowing how many firms they are trying to "outbid" with lower prices, since consumers may or may not search another firm's price.

We will show that efficiency of the equilibrium allocation depends on allowing sellers to announce and commit to reservation prices before the auctions are held. Whereas equilibrium is inefficient when reservation prices are exogenously set to zero, endogenizing seller behavior by allowing sellers to announce and commit to reservation prices results in a unique market equilibrium of the game in which buyers perfectly coordinate, the efficient allocation is achieved, and both allocation and payoffs are equivalent to that of the price-posting case. The contrasting results between exogenous and endogenous reservation prices stems from

[1] Another important feature of the academic job market, and specialized job markets in general, is that firms care exactly *who* they hire for a position. The two-sided heterogeneity we consider is hence appropriate when modeling firms with match-specific preferences for different workers.

[2] This two-stage process results in either Bertrand competition between bidders resulting in workers being offered their marginal productivity, or a monopsony in which the reservation wage is offered to the worker.

the following: when the same side that searches (here, the buyer side) is responsible for price formation, coordination frictions are still present because buyers have incentive to compete over the higher value good and hence could both bid on it. However, when the passive side of the market (here, the seller side) is in effect responsible (here, by setting binding reservation prices), the two-sided heterogeneity results in perfect coordination by the buyers in choosing their goods.

To show this contrast, we will first focus on the case in which reservation prices are exogenously set to zero. This is the most extreme case of buyers–and not sellers–determining equilibrium prices. The assumption that sellers cannot commit to announced reservation prices is appropriate for some markets, especially labor markets in which workers cannot credibly announce to firms that they will not work for higher than a certain wage.

The main result when reservation prices are set to zero is that one buyer–the one with the higher *marginal* valuation for the higher value good–that is, the greater difference between valuations for the two goods–bids on the high value good with probability one, while the other buyer mixes between the two goods when choosing which one to bid on. An immediate consequence of this is that the low value good is either not sold at all or sold for exactly its reservation price. The high value good, on the other hand, is sold for certain, but while it is sold for a price higher than its reservation price with positive probability, this probability is less than one. The buyer with higher marginal value bids exactly the reservation price with positive probability, anticipating that the other buyer may not bid at all. The result is that the potential competition over the high value good does not necessarily result in an equilibrium price higher than the reservation price. Possible ex-post allocations include the socially efficient one in which the buyer with higher marginal value gets the high good and the other buyer gets the low good. However, it is also possible that either buyer obtains the high good while the low good is not sold at all.

After characterizing buyer equilibrium for any reservation prices, we endogenize seller behavior by allowing sellers to commit to reservation prices announced before the buyer game is played. This analysis is in the same spirit as other papers which study seller competition through various mechanisms, such as Peters and Severinov (1997) and McAfee (1993), which is a dynamic version. Burguet and Sakovics (1999) analyze the case with homogenous agents, imperfect information, and second-price auctions for which sellers announce reservation prices to attract buyers. In their model, seller payoffs are continuous in their own strategies and sellers play mixed strategies in all equilibria, none of which are efficient. In contrast, the perfect information framework of our model results in discontinuous seller payoffs in reservation prices. This feature, along with two-sided heterogeneity, results in a unique equilibrium which results in the socially optimal allocation. This result is the same as that of Coles and Eeckhout (2000), who arrive at this result by allowing heterogenous sellers to compete in direct mechanisms for heterogenous buyers. Furthermore, we find that the unique equilibrium when sellers can commit to reservation prices results in the same allocation and payoffs as in the game with price-posting.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 defines and characterizes buyer equilibrium for first the case with zero reservation prices and then for general reservation prices. Section 4 extends the framework by allowing sellers to commit to reservation prices they announce before the buyer game is played. Within this framework, market equilibrium is defined and characterized. Section 5 concludes, while the Appendix includes omitted proofs as well a characterization of buyer equilibrium for the case in which buyers are identical in their marginal valuation of the higher value good.

2 The Model

There are two buyers ($i = 1, 2$) and two sellers ($j = H, L$) each selling one of two distinct goods. For simplicity we will also denote the goods by $j = H, L$. Buyer i's value for good j is publicly known to be v_i^j, with $v_i^H > v_i^L > 0$ for $i = 1, 2$. Seller j's valuation for good j is normalized to zero.

A sealed-bid first-price auction is held for each good simultaneously. In general, the reservation price for good j, denoted by r^j, is chosen by each seller j and announced to the buyers before the buyer game is played.[3] Buyers wish to consume only one good. Furthermore, we assume that buyers are capacity constrained in such a way that they can participate in at most one auction.[4] Each buyer must hence choose the auction in which to participate and submit a bid to only that auction. A crucial assumption is that the auction decision and bid decision must be made simultaneously. Therefore, each buyer participating in an auction must make a bid without knowing whether the other buyer is bidding in the same auction.

Once the buyers have chosen their actions, the highest bidder in each auction is rewarded the auctioned good. If no bidders participated in an auction, neither buyer receives that good. In the event of a tie in either auction, each buyer is rewarded the good with probability one half.[5] All agents are risk neutral. If buyer i is rewarded good j with winning bid b_i^j, his payoff is $v_i^j - b_i^j$. Any buyer who is not rewarded a good receives a payoff of zero. A seller j who sells her good to a buyer bidding b_i^j simply receives payoff b_i^j, while a seller who receive no bids gets a payoff of zero.

3 Buyer Equilibrium

First, we will define an equilibrium of the buyer game. To do so, we will introduce some notation. Denote buyer i's bid on good j by b_i^j. We will consider mixed strategies by the buyers, and so a strategy by buyer i is given by the triplet $(\theta_i, F_i^H(\cdot), F_i^L(\cdot))$, where

$$\begin{aligned}
&\theta_i \text{ is the probability placed by buyer } i \text{ on bidding for good } H,\\
&1-\theta_i \text{ is the probability placed by buyer } i \text{ on bidding for good } L,\\
&F_i^j(\cdot) \text{ is the cdf of the distribution from which buyer } i \text{ chooses } b_i^j.
\end{aligned}$$

Let $\Pi_i^j(b)$ denote the expected payoff for player i from the auction for good j when bidding b, taking the other player's strategy as given. Let $\mathcal{S}_i^j \equiv supp\left[F_i^j(\cdot)\right]$. In any optimal mixed bidding strategy, bidders must be indifferent amongst (almost) all bids in the support of their bidding distributions. We can then write $\Pi_i^j \equiv \Pi_i^j(b)$ for (almost) all $b \in \mathcal{S}_i^j$. The overall expected payoff for buyer i will then be given by $\Pi_i \equiv \theta_i \Pi_i^H + (1-\theta_i)\Pi_i^L$. We can now define our equilibrium concept for the buyer game.

Definition 1 *An equilibrium of the buyer game is a strategy profile* $\left(\theta_i, F_i^j(\cdot)\right)_{i=1,2}^{j=H,L}$, *and scalars* $\left(\Pi_i^j\right)_{i=1,2}^{j=H,L}$ *where* $\theta_i \in [0,1]$ *and* F_i^j*s are distribution functions such that, for both* $i = 1, 2$:

[3] However, we will not extend the framework to allow for endogenous reservation prices until section 4.

[4] A motivating example for this assumption is a labor market in which firms (buyers) with one job opening cannot feasibly offer a job to two workers (sellers) who could potentially both accept. A possible interpretation is that the disutility for a firm from having to back out of a commitment it cannot meet is sufficiently for it to simply never take the risk of "winning" two workers (goods) by placing multiple bids.

[5] This tiebreaking rule is arbitrary and does not play a role in any results.

1. $\Pi_i^j = \Pi_i^j(b)$ *for almost all* $b \in \mathcal{S}_i^j$ *for both* j,

2. $\Pi_i^j \geq \Pi_i^j(b)$ *for all* b *for both* j,

3. $\theta_i \in \arg\max_{\theta} \theta\Pi_i^H + (1-\theta)\Pi_i^L$.

Section 3.1 focuses on the buyer equilibrium when reservation prices are zero. Section 3.2 then generalizes this result to any pair of reservation prices r^H and r^L.

3.1 Buyer equilibrium when reservation prices are zero

We will first provide some necessary conditions for equilibrium when reservation prices are identical and exogenously set equal to zero. We will derive the unique buyer equilibrium for generic parameter values for this case.

Neither buyer will leave the higher-valued good uncontested with probability one. It is somewhat intuitive to suggest that neither player will be willing to give up on the high valued good for certain. Doing so would essentially allow the other buyer to have it for free, which will result in the immediate incentive for the first player to deviate to bidding an arbitrarily small amount for the high value good and obtaining it after all. Our first claim formally states such an equilibrium does not exist.

Claim 1 *There is no equilibrium in which either bidder bids on good* L *with probability one. (It cannot hold that* $\theta_i = 0$ *for either* $i = 1, 2$.*)*
Proof. *Suppose* $\theta_1 = 0$. *Then buyer* 2*'s best response is to play the pure strategy* $(\theta_2 = 1, b_2^H = 0)$, *bidding zero on good* H, *getting his maximum possible payoff of* $\Pi_2 = v_2^H$. *Buyer* 1*'s resulting payoff would then be* $\Pi_1 = v_1^L$. *But then buyer* 1 *could deviate to bid* $\varepsilon > 0$ *on good* H, *getting a payoff of* $v_1^H - \varepsilon > v_1^L$ *for* ε *sufficiently small.* ■

Both buyers will never bid on the same good with probability one. We have the following result for when there is only one auction with two bidders.

Lemma 1 *Suppose both bidders are bidding in the auction for good* j *with probability one and that* $v_1^j \geq v_2^j$. *Then the only equilibria are ones in which buyer* 1 *places probability one on some* $\bar{b} \in [v_2^j, v_1^j)$ *and buyer* 2 *plays a mixed bidding strategy satisfying (i)*$\lim_{b \nearrow \bar{b}} F_2^j(b) = 1$, *(ii)*$F_2^j(b) \leq \frac{v_1^j - \bar{b}}{v_1^j - b}$ *for all* $b \leq \bar{b}$. *Such equilibria result in the payoffs* $\Pi_1^j = v_1^j - \bar{b}$, $\Pi_2^j = 0$. *If we make the further restriction that buyers must play only undominated strategies, it must hold that* $\bar{b} = v_2^j$.
Proof. *In Appendix.* ■

Because there are, in pure quantity, enough goods to go around, no buyer will ever receive a zero expected payoff in any buyer equilibrium. From lemma 1, it is hence immediate that both buyers will not both bid on good H with probability one.

Claim 2 *There is no equilibrium in which both buyers bid on good* H *with probability one. (It cannot hold that* $\theta_1 = \theta_2 = 1$.*)*
Proof. *Suppose* $\theta_1 = \theta_2 = 1$. *Without loss of generality, suppose* $v_1^H \geq v_2^H$. *From lemma 1 we immediately have that* $\Pi_2^H = 0$ *in any equilibrium of the auction for good* H. *But then any deviation by buyer* 2 *in which he bids some* $b_2^L < v_2^L$ *for good* L *will yield a positive payoff and is hence profitable.* ■

No equilibria generically with both buyers mixing between auctions For the rest of this section, we will consider cases in which both buyers place positive probability on both auctions. We will show that no such case can be an equilibrium for any but a zero measure subset of the parameter space.

First, we will show that both buyers must mix in their bidding strategy in a particular auction, and that the mixing distributions from which they select their bids must be continuous at all positive bids.

Lemma 2 *Suppose both buyers place positive probability on bidding for good j, with at least one buyer placing less than probability one. Then, it must be that neither buyer plays any positive bid with positive probability. (That is, there exists no $\widehat{b} > 0$ such that $\lim_{b \nearrow \widehat{b}} F_i^j(b) \neq F_i^j(\widehat{b})$ for either $i = 1, 2$.)*
Proof. *In Appendix.* ■

Lemma 3 *Suppose both buyers place positive probability on bidding for good j, with at least one buyer placing less than probability one. Then it must hold that the supports $\mathcal{S}_i^j$ are convex for both $i = 1, 2$.*
Proof. *In Appendix.* ■

Now, we will show that $\mathcal{S}_1^j$ and $\mathcal{S}_2^j$ must have the same maximum point and share a minimum point of zero.

Lemma 4 *Suppose both buyers place positive probability on bidding for good j, with at least one buyer placing less than probability one. Then it must hold that: (i) The maximum possible bid placed on good j by buyer 1 must be the same as the maximum possible bid placed on good j by buyer 2. That is, $\overline{b_1^j} = \overline{b_2^j}$ ($\equiv \overline{b^j}$). (ii) The minimum possible bid placed on good j must be zero for both buyers. That is, $\min \mathcal{S}_i^j = 0$ for both $i = 1, 2$.*
Proof. *In Appendix.* ■

Next, we will show that when both buyers have a positive probability of participating in an auction, at least one with probability less than one, the payoff for a buyer in that auction can be written as the difference between his value and the maximum possible bid both buyers place on the good.

Lemma 5 *Suppose both buyers place positive probability on bidding for good j, with at least one buyer placing less than probability one. Then the expected payoff for a buyer from the auction for good j is the buyer's value of good j minus the common maximum of the supports of the distributions from which the buyers choose their bids for good j. That is, $\Pi_i^j = v_i^j - \overline{b^j}$.*
Proof. *In Appendix.* ■

We can now state that there is no equilibrium in which both players mix between the two auctions.

Claim 3 *There is no equilibrium in which both players mix between the two auctions, other than for a zero measure of the parameter space. In particular, such an equilibrium only exists when $v_1^H - v_1^L = v_2^H - v_2^L$.*
Proof. *For both players to be mixing between the two auctions, it must hold that $\Pi_i^H = \Pi_i^L$ for both $i = 1, 2$. Lemma 5 states that buyer i's payoff when bidding $\overline{b^j}$ for good j is given by $\Pi_i^j(\overline{b^j}) = (v_i^j - \overline{b^j})$. But then we have as a necessary condition that $v_i^H - \overline{b^H} = v_i^L - \overline{b^L}$ for both $i = 1, 2$, which can only hold if $v_1^H - v_1^L = v_2^H - v_2^L$.* ■

See **Figure 1** for a graphic description of the result stated in Claim 3.

The unique buyer equilibrium To summarize our results so far, we have shown that there are (1) no equilibria in which either player bids on the low value good for certain, (2) no equilibria which both players bid in the same auction for certain, and (3) no equilibria (for a generic set of parameters) in which both buyers mix between both auctions.

The force behind the first two restrictions involves the incentive for buyers to infinitesimally outbid their opponent when a good is left "uncontested" in a candidate equilibria. The third restriction arises due to the impossibility of both buyers being exactly indifferent between bidding in the two auctions. This impossibility stems from the buyers' (generically) different valuations for the goods, combined with the equilibrium condition that the difference between their expected payoffs from a given auction can be pinned down by their values.

In this section we will show that this game has a unique equilibrium for a generic set of parameters. This equilibrium will involve the only possible combination of auction choices we have not already ruled out: one in which one buyer mixes between the two auctions and the other buyer bids on the high value good with probability one. The buyer who bids on the high value good for certain must be the one who has the higher "gap" between his two valuations, while the "low-gap" buyer will mix between attempting to win the high value good through a mixed bidding strategy and simply taking the low value good by bidding zero and obtaining his full value. Without loss of generality, we can assume $v_1^H - v_1^L \geq v_2^H - v_2^L$ and let buyer 1 take on the role of the "high-gap" buyer. To establish the unique equilibrium of the game, we first need one more lemma.

Lemma 6 *It cannot hold in any equilibrium that both players bid zero on the same good each with positive probability. (That is, it cannot hold that both $F_1^j(0) > 0$ and $F_2^j(0) > 0$ for either $j = H, L$.)*
Proof. *In Appendix.* ■

We can now characterize the unique buyer equilibrium for generic parameters. See **Figure 2** for a graphic description of this buyer equilibrium.

Theorem 1 *Let $v_1^H - v_1^L > v_2^H - v_2^L$. Then the unique equilibrium of the game is the one in which buyer 1 bids on the high value good with probability one, buyer 2 mixes between bidding in the auctions, always bidding zero on the low value good, and both buyers mix in their bidding for the high value good in such a way that both are indifferent between all bids in the support of their own distribution, neither has a bid outside his support which yields a higher payoff, and buyer 2 is exactly indifferent between bidding in the two auctions. Such mixing involves buyer 1 placing positive probability on bidding zero for the high value* *good. In particular, this equilibrium is given by*

$$
\begin{aligned}
\theta_1 &= 1, \\
F_1^H(b) &= \begin{cases} \frac{v_2^L}{v_2^H - b} & \text{for } b \in \left[0, v_2^H - v_2^L\right], \\ 1 & \text{for } b > v_2^H - v_2^L, \end{cases} \\
\theta_2 &= \frac{v_2^H - v_2^L}{v_1^H}, \\
F_2^H(b) &= \begin{cases} \left(\frac{v_1^H - v_2^H + v_2^L}{v_2^H - v_2^L}\right) \cdot \frac{b}{v_1^H - b} & \text{for } b \in \left[0, v_2^H - v_2^L\right], \\ 1 & \text{for } b > v_2^H - v_2^L, \end{cases} \\
F_2^L(b) &= 1 \text{ for } b \geq 0, \\
\Pi_1 &= \Pi_1^H = v_1^H - \left(v_2^H - v_2^L\right), \\
\Pi_2 &= \Pi_2^H = \Pi_2^L = v_2^L.
\end{aligned}
$$

Proof. *It is straightforward to check that this is indeed an equilibrium of the game. Buyer 1 is getting a payoff of $\Pi_1 = v_1^H - \left(v_2^H - v_2^L\right)$ from all bids in his support $\left[0, v_2^H - v_2^L\right]$, which we can demonstrate using the fact that in general buyer 1's payoff from the auction for good H as a function of his bid can be written as $\Pi_1^H(b) = \left(v_1^H - b\right)\left[\theta_2 F_2^H(b) + (1 - \theta_2)\right]$. Simply substituting the equilibrium objects for $b \in \left[0, v_2^H - v_2^L\right]$ yields*

$$
\begin{aligned}
\Pi_1^H(b) &= \left(v_1^H - b\right)\left[\theta_2 F_2^H(b) + (1 - \theta_2)\right] \\
&= \left(v_1^H - b\right)\left[\frac{v_2^H - v_2^L}{v_1^H} \cdot \left(\left(\frac{v_1^H - v_2^H + v_2^L}{v_2^H - v_2^L}\right) \cdot \frac{b}{v_1^H - b}\right) + \left(1 - \frac{v_2^H - v_2^L}{v_1^H}\right)\right] \\
&= v_1^H - \left(v_2^H - v_2^L\right).
\end{aligned}
$$

Deviating to any bid $b' > v_2^H - v_2^L$ will result in a payoff of $v_1^H - b' < \Pi_1$, while bidding arbitrarily close to zero to obtain the low value good will result in a payoff which approaches v_1^L, and by assumption we have that $v_1^L \leq v_1^H - \left(v_2^H - v_2^L\right) = \Pi_1$. Buyer 2 is receiving a payoff of $\Pi_2^j = v_2^L$ for both auctions $j = H, L$. In the auction for good L, he is simply bidding zero and obtaining his value v_2^L uncontested. In the auction for good H, his payoff can be written as $\Pi_2^H(b) = \left(v_2^H - b\right)\left[\theta_1 F_1^H(b) + (1 - \theta_1)\right]$, which when substituting the equilibrium objects for $b \in \left[0, v_2^H - v_2^L\right]$ yields

$$
\begin{aligned}
\Pi_2^H(b) &= \left(v_2^H - b\right)\left[\theta_1 F_1^H(b) + (1 - \theta_1)\right] \\
&= \left(v_2^H - b\right) \cdot \left[1 \cdot \frac{v_2^L}{v_2^H - b} + (1 - 1)\right] \\
&= v_2^L.
\end{aligned}
$$

Deviating to any bid $b' > v_2^H - v_2^L$ *will result in a payoff of* $v_2^H - b' < v_2^L = \Pi_2$*. So we have established that the above strategy profile is in fact an equilibrium. To prove uniqueness, we rely on claims 1,2, and 3 to first rule out all auction choice behavior possibilities other than one buyer mixing between the two auctions and the other bidding on the high value good with probability one. Arbitrarily calling the mixing buyer "buyer 2", we know that, being uncontested in the auction for the low value good, he must be bidding* $b_2^L = 0$ *with probability one and hence receiving* $\Pi_2^L = v_2^L$*. From lemma 5 we further know that payoffs from the high value auction must be given by* $\Pi_i^H = v_i^H - \overline{b^H}$*. Since buyer 2 must be indifferent between the two auctions, and buyer 1 must be unable to achieve a higher payoff than* $v_1^H - \overline{b^H}$ *by bidding an arbitrarily small amount for the low value good, we have the necessary conditions* $v_2^H - \overline{b^H} = v_2^L$ *and* $v_1^H - \overline{b^H} \geq v_1^L$*, which yields the condition* $v_1^H - v_1^L \geq v_2^H - v_2^L$*. That is, the buyer mixing between the two auctions must be what we call the "low-gap" buyer. The first condition also pins down* $\overline{b^H} = v_2^H - v_2^L$*. From part (ii) of lemma 4, we know that* $\min \mathcal{S}_i^H = 0$ *for both* $i = 1, 2$ *and from lemma 3 we know that* $\mathcal{S}_i^H$ *is convex for both* $i = 1, 2$*, which yields* $\mathcal{S}_1^H = \mathcal{S}_2^H = \left[0, v_2^H - v_2^L\right]$*. Also, lemma 2 tells us that* F_i^H *cannot have a discontinuity at any positive bid for either* $i = 1, 2$*. Analyzing the payoff for player* 2 *from bids at the bottom of the support* $\left[0, v_2^H - v_2^L\right]$*, we have, since* $\theta_1 = 1$ *has been established, that*

$$\begin{aligned}
\Pi_2^H &= \lim_{\varepsilon \to 0} (v_2^H - \varepsilon) \left[\theta_1 F_1^H(\varepsilon) + (1 - \theta_1)\right] \\
&= \lim_{\varepsilon \to 0} (v_2^H - \varepsilon) F_1^H(\varepsilon) \\
&= v_2^H \cdot \lim_{\varepsilon \to 0} F_1^H(\varepsilon),
\end{aligned}$$

and so for $\Pi_2^H > 0$ *it must hold that* $\lim_{\varepsilon \to 0} F_1^H(\varepsilon) = F_1^H(0) > 0$*. In particular, since* $\Pi_2^H = \Pi_2^L = v_2^L$ *has been established, we have that* $v_2^L = v_2^H \cdot F_1^H(0)$*, or* $F_1^H(0) = v_2^L / v_2^H$*. From lemma 6, we have that at most one of* F_1^H *and* F_2^H *can have a discontinuity at zero, and hence* $F_2^H(0) = 0$*. Then, analyzing the payoff for player* 1 *when bidding zero yields*

$$\begin{aligned}
\Pi_1^H &= \Pi_1^H(0) \\
&= (v_1^H - 0) \left[\theta_2 \underbrace{F_2^H(0)}_{=0} + (1 - \theta_2)\right] \\
&= v_1^H (1 - \theta_2),
\end{aligned}$$

which, along with the already established condition $\Pi_1^H = v_1^H - \overline{b^H} = v_1^H - \left(v_2^H - v_2^L\right)$*, yields* $v_1^H - \left(v_2^H - v_2^L\right) = v_1^H (1 - \theta_2)$*, or* $\theta_2 = (v_2^H - v_2^L)/v_1^H$*. The exact distributions* F_1^H *and* F_2^H *are then uniquely pinned down uniquely by the indifference conditions*

$$v_2^L = \Pi_2^H = \Pi_2^H(b) = (v_2^H - b) \left[\underbrace{\theta_1}_{=1} F_1^H(b) + \left(1 - \underbrace{\theta_1}_{=1}\right)\right] \text{ for all } b \in \left[0, v_2^H - v_2^L\right]$$

and

$$v_1^H - (v_2^H - v_2^L) = \Pi_1^H = \Pi_1^H(b) = (v_1^H - b) \left[\underbrace{\theta_2}_{=(v_2^H - v_2^L)/v_1^H} F_2^H(b) + \left(1 - \underbrace{\theta_2}_{=(v_2^H - v_2^L)/v_1^H}\right)\right] \text{ for all } b \in \left[0, v_2^H - v_2^L\right],$$

respectively. ■

3.2 Buyer Equilibrium for General Reservation Prices

The results of the previous subsection are driven by two crucial, if somewhat implicit, assumptions. First, the two buyers are assumed to both prefer the same good (good H) should they be allowed to receive their good of choice without competition from the other buyer. Secondly, reservation prices are low enough that neither buyer is "priced out" of bidding for either good. That is, the reservation price for each good is lower than both buyers' valuations for that good.

In this subsection, we will derive the buyer equilibrium for any reservation prices $(r^H, r^L) \in \mathbb{R}^2_+$. In doing so, we will consider reservation prices for which one or both of the implicit assumptions above are violated. The first assumption will be violated for buyer i whenever $r^H - r^L > v_i^H - v_i^L$. Note that for such reservation prices, buyer i would choose to purchase good L for price r^L rather than good H for price r^H when given the choice to choose a good uncontested by the other buyer. The second assumption will be violated for a buyer i bidding on good j whenever $r^j > v_i^j$. Such a buyer will be priced out of bidding for good j.

It turns out that there are eight significant regions of the reservation price space $\mathbb{R}^2_+$ when analyzing buyer equilibrium. Each of these eight regions has a qualitatively different type of buyer equilibrium associated with it, and it holds that all reservation price pairs $(r^H, r^L) \in \mathbb{R}^2_+$ have either a unique buyer equilibrium associated with it or have multiple equilibria which are all payoff equivalent for the buyers when restricting attention to undominated strategies. Along the boundaries between the eight regions, there are multiple buyer equilibria in undominated strategies, but in these cases all undominated strategy equilibria are again payoff equivalent for the buyers. This result stems directly from the fact that equilibrium buyer payoffs are continuous in reservation prices, even across the different regions. We will now describe the eight regions and the buyer equilibria associated with each one. It helps to introduce the following notation: let $i^{\max j} \equiv \underset{i \in \{1,2\}}{\arg\max}\, v_i^j$ and $i^{\min j} \equiv \underset{i \in \{1,2\}}{\arg\min}\, v_i^j$. Also, see **Figures 3.1,3.2 and 3.3** for a graphic interpretation on these eight regions.

3.2.1 Region $1_H 2_{mix}$

The region $1_H 2_{mix}$ is defined by

$$1_H 2_{mix} \equiv \left\{ (r^H, r^L) \in \mathbb{R}^2_+ : r^H - r^L \leq v_2^H - v_2^L, r^L \leq \min\{v_2^L, v_2^L - (v_2^H - v_1^H)\} \right\}.$$

This region contains the origin and is named to describe the type of buyer equilibrium we derived in section 3.1. The region is the one in which both buyers prefer good H to good L when uncontested, buyer 2 is not priced out of bidding for either good, buyer 1 is not priced out of bidding for good H, and buyer 2 is able to make a sufficiently high payoff bidding on L that he is willing to mix.[6]

[6] This last condition is violated when $r^L > v_2^L - (v_2^H - v_1^H)$, which is a binding constraint for (r^H, r^L) belonging to $1_H 2_{mix}$ if and only if $v_2^H > v_1^H$.

Proposition 1 *The unique buyer equilibrium for any reservation prices* $(r^H, r^L) \in \operatorname{int} 1_H 2_{mix}$ *is given by*

$$\theta_1 = 1,$$

$$F_1^H(b) = \begin{cases} 0 & \text{for } b < r^H, \\ \frac{v_2^L - r^L}{v_2^H - b} & \text{for } b \in \left[r^H, v_2^H - v_2^L + r^L\right], \\ 1 & \text{for } b > v_2^H - v_2^L + r^L, \end{cases}$$

$$\theta_2 = \frac{v_2^H - v_2^L - (r^H - r^L)}{v_1^H - r^H},$$

$$F_2^H(b) = \begin{cases} 0 & \text{for } b < r^H, \\ \left(\frac{v_1^H - v_2^H + v_2^L - r^L}{v_2^H - v_2^L - (r^H - r^L)}\right) \cdot \frac{b - r^H}{v_1^H - b} & \text{for } b \in \left[r^H, v_2^H - v_2^L + r^L\right], \\ 1 & \text{for } b > v_2^H - v_2^L + r^L, \end{cases}$$

$$F_2^L(b) = \begin{cases} 0 & \text{for } b < r^L, \\ 1 & \text{for } b \geq r^L, \end{cases}$$

$$\Pi_1 = \Pi_1^H = v_1^H - v_2^H + v_2^L - r^L,$$

$$\Pi_2 = \Pi_2^H = \Pi_2^L = v_2^L - r^L.$$

Proof. *The proof is identical to that of Theorem 1, with the reservation prices* r^H *and* r^L *trivially replacing zero in the calculations.* ■

3.2.2 Region $2_L 1_{mix}$

The region $2_L 1_{mix}$ is defined by

$$2_L 1_{mix} \equiv \left\{(r^H, r^L) \in \mathbb{R}_+^2 : r^H - r^L \geq v_1^H - v_1^L, r^H \leq \min\{v_1^H, v_1^H - (v_1^L - v_2^L)\}\right\}.$$

This region is the one in which both buyers prefer good L to good H when uncontested, buyer 1 is not priced out of bidding for either good, buyer 2 is not priced out of bidding for good L, and buyer 1 is able to make a sufficiently high payoff bidding on H that he is willing to mix.[7] In buyer equilibria associated with these reservation prices, the roles of goods H and L are reversed from those in region $1_H 2_{mix}$, and hence the roles of buyers 1 and 2 are also reversed; in both regions, it is the buyer with the lower marginal valuation for the *preferred* good which mixes between goods, where by preferred good we mean the one which would be chosen given the reservation prices if the other buyer did not exist. The buyer with the lower marginal

[7] This last condition is violated when $r^H > v_1^H - (v_1^L - v_2^L)$, which is a binding constraint for (r^H, r^L) belonging to $2_L 1_{mix}$ if and only if $v_1^L > v_2^L$.

valuation for the preferred good in region $2_L 1_{mix}$ is indeed buyer 1, whose marginal valuation for good L can be written as $(v_1^L - r^L) - (v_1^H - r^H)$, which is in fact greater than zero but less than $(v_2^L - r^L) - (v_2^H - r^H)$, the marginal valuation for good L of buyer 2. The buyer equilibria in this region, not surprisingly, looks exactly like the ones in region $1_H 2_{mix}$, with only the names of the goods and buyers reversed.

Proposition 2 *The unique buyer equilibrium for any reservation prices* $(r^H, r^L) \in \operatorname{int} 2_L 1_{mix}$ *is given by*

$$\theta_1 = \frac{v_2^L - v_1^L + v_1^H - r^H}{v_2^L - r^L},$$

$$F_1^H(b) = \begin{cases} 0 & \textit{for } b < r^H, \\ 1 & \textit{for } b \geq r^H, \end{cases}$$

$$F_1^L(b) = \begin{cases} 0 & \textit{for } b < r^L, \\ \left(\frac{v_2^L - v_1^L + v_1^H - r^H}{v_1^L - v_1^H - (r^L - r^H)}\right) \cdot \frac{b - r^L}{v_2^L - b} & \textit{for } b \in \left[r^L, v_1^L - v_1^H + r^H\right], \\ 1 & \textit{for } b > v_1^L - v_1^H + r^H, \end{cases}$$

$$\theta_2 = 0,$$

$$F_2^L(b) = \begin{cases} 0 & \textit{for } b < r^L, \\ \frac{v_1^H - r^H}{v_1^L - b} & \textit{for } b \in \left[r^L, v_1^L - v_1^H + r^H\right], \\ 1 & \textit{for } b > v_1^L - v_1^H + r^H, \end{cases}$$

$$\Pi_1 = \Pi_1^H = \Pi_1^L = v_1^H - r^H,$$

$$\Pi_2 = \Pi_2^L = v_2^L - v_1^L + v_1^H - r^H.$$

Proof. *The proof is identical to that of Theorem 1, with the reservation prices* r^H *and* r^L *trivially replacing zero in the calculations and the names of the goods and buyers reversed.* ■

3.2.3 Region $1_H 2_L$

The region $1_H 2_L$ is defined by

$$1_H 2_L \equiv \left\{(r^H, r^L) \in \mathbb{R}_+^2 : v_2^H - v_2^L \leq r^H - r^L \leq v_1^H - v_1^L, r^H \leq v_1^H, r^L \leq v_2^L\right\}.$$

This region is the one in which, when each buyer is uncontested, buyer 1 prefers good H while buyer 2 prefers good L, and in which neither buyer is priced out of bidding for his preferred good. As one would expect, since the buyers prefer different goods, there is perfect coordination in the unique buyer equilibrium associated with each pair (r^H, r^L) in this region.

Proposition 3 *The unique buyer equilibrium for any reservation prices* $(r^H, r^L) \in \operatorname{int} 1_H 2_L$ *is given by*

$$\begin{aligned}
\theta_1 &= 1, \\
F_1^H(b) &= \begin{cases} 0 & \text{for } b < r^H, \\ 1 & \text{for } b \geq r^H, \end{cases} \\
\theta_2 &= 0, \\
F_2^L(b) &= \begin{cases} 0 & \text{for } b < r^L, \\ 1 & \text{for } b \geq r^L, \end{cases} \\
\Pi_1 &= \Pi_1^H = v_1^H - r^H, \\
\Pi_2 &= \Pi_2^L = v_2^L - r^L.
\end{aligned}$$

Proof. *It is immediate that the strategy profile described is a buyer equilibrium, since both buyers are obtaining the highest feasible payoff given reservation prices* (r^H, r^L)*. To show that there are no other equilibria, suppose* $\theta_1 < 1$*.If* $\theta_2 = 1$*, then buyer 1 must be bidding* $b_1^L = r^L$ *on good* L *with probability one, allowing for buyer 2 to make a profitable deviation of bidding* $r^L + \varepsilon$ *for* $\varepsilon > 0$ *arbitrarily small on good* L*. So it must be that* $\theta_2 < 1$*. Now, if* $\theta_1 = 0$*, then buyer 2 must be bidding* $b_2^H = r^H$ *on good* H *with probability one, allowing for buyer 1 to make a profitable deviation of bidding* $r^H + \varepsilon$ *for* $\varepsilon > 0$ *arbitrarily small on good* H*. So it must then hold that* $\theta_1 \in (0, 1)$*. But if* $\theta_2 = 0$*, then buyer 1 is strictly better off bidding* r^H *on good* H *than bidding any bid on good* L*, so it must also hold that* $\theta_2 \in (0, 1)$*. However, for both buyers to be indifferent between both goods, it must hold that* $v_1^H - v_1^L = v_2^H - v_2^L$*, a zero measure subset of the parameter space in which region* $1_H 2_L$ *also has zero measure and* $\operatorname{int} 1_H 2_L = \varnothing$*.* ■

3.2.4 Region $1_H 2_H$

The region $1_H 2_H$ is defined by

$$1_H 2_H \equiv \left\{ (r^H, r^L) \in \mathbb{R}_+^2 : r^H \leq \min\{v_1^H, v_2^H\}, r^L \geq \min\{v_2^L, v_2^L - (v_2^H - v_1^H)\} \right\}.$$

This region is the one in which both buyers prefer good H when uncontested, buyer 2 is either priced out of bidding for good L or cannot obtain a high enough payoff bidding on good L to make mixing between L and H worthwhile, and neither buyer is priced out of bidding on good H. The buyer with the higher valuation for good H obtains good H, while the buyer with the lower valuation does not obtain either good and serves solely to discipline the price of good H. In undominated strategies, the equilibrium price of good H is necessarily $v_{i^{\min H}}^H$.

Proposition 4 *Any buyer equilibrium in undominated strategies for any reservation prices* $(r^H, r^L) \in$

$\operatorname{int} 1_H 2_H$ *must satisfy*

$$
\begin{aligned}
\theta_1 &= 1, \\
F^H_{i^{\max H}}(b) &= \begin{cases} 0 & \textit{for } b < v^H_{i^{\max H}} - v^H_{i^{\min H}}, \\ 1 & \textit{for } b \geq v^H_{i^{\max H}} - v^H_{i^{\min H}}, \end{cases} \\
\theta_2 &= 1, \\
F^H_{i^{\min H}}(b) & \begin{cases} \leq \frac{v^H_{i^{\max H}} - v^H_{i^{\min H}}}{v^H_{i^{\max H}} - b} & \textit{for } b < v^H_{i^{\min H}}, \\ = 1 & \textit{for } b \geq v^H_{i^{\min H}}, \end{cases} \\
\Pi_{i^{\max H}} &= \Pi^H_{i^{\max H}} = v^H_{i^{\max H}} - v^H_{i^{\min H}}, \\
\Pi_{i^{\min H}} &= \Pi^H_{i^{\min H}} = 0.
\end{aligned}
$$

Proof. *Note first that the conditions defining region $1_H 2_H$ imply that $r^H - r^L \leq v^H_2 - v^L_2$. Now first suppose $v^H_1 \geq v^H_2$. Then $r^L > v^L_2$ and so buyer 2 is priced out of bidding for good L. Therefore $\theta_2 = 1$. By lemma 1, buyer 1 will win good H and receive a payoff of $v^H_1 - v^H_2$ in any equilibrium in undominated strategies in which $\theta_1 = \theta_2 = 1$. If buyer 1 were to bid on good L instead, he would receive a payoff of $v^L_1 - r^L \leq v^L_1 - v^L_2 \leq v^L_1 - (v^L_1 - v^H_1 + v^H_2) = v^H_1 - v^H_2$. Generically, the second inequality holds strictly. So $\theta_1 = 1$ and by lemma 1 we get the conditions for F^H_1 and F^H_2. Next, suppose $v^H_2 \geq v^H_1$. Then $r^L > v^L_2 - (v^H_2 - v^H_1)$ and $r^L > v^L_1$, so buyer 1 is priced out of bidding for good L, meaning $\theta_1 = 1$. Again using lemma 1, buyer 2 will receive good H and receive payoff $v^H_2 - v^H_1$ in any equilibrium in undominated strategies for which $\theta_1 = \theta_2 = 1$, and bidding on good L instead yields buyer 2 a payoff of $v^L_2 - r^L \leq v^L_2 - (v^L_2 - v^H_2 + v^H_1) = v^H_2 - v^H_1$. So $\theta_2 = 1$ and the conditions on F^H_2 and F^H_1 follow from lemma 1.* ■

3.2.5 Region $1_L 2_L$

The region $1_L 2_L$ is defined by

$$
1_L 2_L \equiv \left\{ (r^H, r^L) \in \mathbb{R}^2_+ : r^H \geq \min\{v^H_1, v^H_1 - (v^L_1 - v^L_2)\}, r^L \leq \min\{v^L_1, v^L_2\} \right\}.
$$

This region is the one in which both buyers prefer good L when uncontested, buyer 1 is either priced out of bidding for good H or cannot obtain a high enough payoff bidding on good H to make mixing between H and L worthwhile, and neither buyer is priced out of bidding on good L. The buyer with the higher valuation for good L obtains good L, while the buyer with the lower valuation does not obtain either good and serves solely to discipline the price of good L. In undominated strategies, the equilibrium price of good L is necessarily $v^L_{i^{\min L}}$.

Proposition 5 *Any buyer equilibrium in undominated strategies for any reservation prices* $(r^H, r^L) \in \operatorname{int} 1_L 2_L$ *must satisfy*

$$\begin{aligned}
\theta_1 &= 0, \\
F^L_{i^{\max L}}(b) &= \begin{cases} 0 & \text{for } b < v^L_{i^{\max L}} - v^L_{i^{\min L}}, \\ 1 & \text{for } b \geq v^L_{i^{\max L}} - v^L_{i^{\min L}}, \end{cases} \\
\theta_2 &= 0, \\
F^L_{i^{\min L}}(b) & \begin{cases} \leq \frac{v^L_{i^{\max L}} - v^L_{i^{\min L}}}{v^L_{i^{\max L}} - b} & \text{for } b < v^L_{i^{\min L}}, \\ = 1 & \text{for } b \geq v^L_{i^{\min L}}, \end{cases} \\
\Pi_{i^{\max L}} &= \Pi^L_{i^{\max L}} = v^L_{i^{\max L}} - v^L_{i^{\min L}}, \\
\Pi_{i^{\min L}} &= \Pi^L_{i^{\min L}} = 0.
\end{aligned}$$

Proof. *The proof is identical to that of proposition 4, with only the names of the goods and buyers reversed.* ■

3.2.6 Region $i_H^{\max H} i_O^{\min H}$

The region $i_H^{\max H} i_O^{\min H}$ is defined by

$$i_H^{\max H} i_O^{\min H} \equiv \left\{ (r^H, r^L) \in \mathbb{R}^2_+ : r^H - r^L \leq v^H_{i^{\max H}} - v^L_{i^{\max H}}, v^H_{i^{\min H}} \leq r^H \leq v^H_{i^{\max H}}, r^L \geq v^L_{i^{\min H}} \right\}.$$

This region is the one in which buyer $i^{\max H}$ prefers good H to good L when uncontested and is not priced out of bidding for good H, while buyer $i^{\min H}$ is priced out of bidding for either good. Trivially, we get the result that buyer $i^{\max H}$ will receive good H and a payoff of $v^H_{i^{\max H}} - r^H$, while buyer $i^{\min H}$ will not participate and receive a payoff of zero.

Proposition 6 *Any buyer equilibrium in undominated strategies for any reservation prices* $(r^H, r^L) \in$

$\operatorname{int} i_H^{\max H} i_O^{\min H}$ *must satisfy*

$$\begin{aligned}
\theta_{i^{\max H}} &= 1, \\
F^H_{i^{\max H}}(b) &= \begin{cases} 0 & \textit{for } b < r^H, \\ 1 & \textit{for } b \geq r^H, \end{cases} \\
\lim_{b \nearrow r^H} F^H_{i^{\min H}}(b) &= 1 \\
\lim_{b \nearrow r^L} F^L_{i^{\min H}}(b) &= 1 \\
\Pi_{i^{\max H}} &= \Pi^H_{i^{\max H}} = v^H_{i^{\max H}} - r^H, \\
\Pi_{i^{\min H}} &= 0.
\end{aligned}$$

Proof. *It is immediate that buyer $i^{\min H}$ will not submit a bid higher than $v^j_{i^{\min H}}$ to either auction j in undominated strategies. Since $v^j_{i^{\min H}} < r^j$ for both goods j and because $r^H - r^L < v^H_{i^{\max H}} - v^L_{i^{\max H}}$, buyer $i^{\max H}$ prefers good H and simply bids the reservation price r^H and wins for sure.* ■

3.2.7 Region $i_L^{\max L} i_O^{\min L}$

The region $i_L^{\max L} i_O^{\min L}$ is defined by

$$i_L^{\max L} i_O^{\min L} \equiv \left\{ (r^H, r^L) \in \mathbb{R}^2_+ : r^H - r^L \geq v^H_{i^{\max L}} - v^L_{i^{\max L}}, v^L_{i^{\min L}} \leq r^L \leq v^L_{i^{\max L}}, r^H \geq v^H_{i^{\min L}} \right\}.$$

This region is the one in which buyer $i^{\max L}$ prefers good L to good H when uncontested and is not priced out of bidding for good L, while buyer $i^{\min L}$ is priced out of bidding for either good. Trivially, we get the result that buyer $i^{\max L}$ will receive good L and a payoff of $v^L_{i^{\max L}} - r^L$, while buyer $i^{\min L}$ will not participate and receive a payoff of zero.

Proposition 7 *Any buyer equilibrium in undominated strategies for any reservation prices $(r^H, r^L) \in$*

int $i_L^{\max L} i_O^{\min L}$ *must satisfy*

$$\theta_{i^{\max L}} = 0,$$

$$F^L_{i^{\max L}}(b) = \begin{cases} 0 & \textit{for } b < r^L, \\ 1 & \textit{for } b \geq r^L, \end{cases}$$

$$\lim_{b \nearrow r^H} F^H_{i^{\min L}}(b) = 1$$

$$\lim_{b \nearrow r^L} F^L_{i^{\min L}}(b) = 1$$

$$\Pi_{i^{\max L}} = \Pi^L_{i^{\max L}} = v^L_{i^{\max L}} - r^L,$$

$$\Pi_{i^{\min L}} = 0.$$

Proof. *The proof is identical to that of proposition 6, with only the names of the goods reversed.* ■

3.2.8 Region $1_O 2_O$

The region $1_O 2_O$ is defined by

$$1_O 2_O \equiv \left\{(r^H, r^L) \in \mathbb{R}^2_+ : r^H \geq \max\{v^H_1, v^H_2\}, r^L \geq \max\{v^L_1, v^L_2\}\right\}.$$

This is the region in which both buyers are priced out of bidding for goods, and so trivially we get that the only buyer equilibria are ones in which neither buyer submits an acceptable bid.

Proposition 8 *Any buyer equilibrium in undominated strategies for any reservation prices* $(r^H, r^L) \in$ int $1_O 2_O$ *must satisfy, for both* $i = 1, 2$ *and* $j = H, L$,

$$\lim_{b \nearrow r^j} F^j_i(b) = 1,$$

$$\Pi_i = 0.$$

Proof. *Both buyers are priced out of bidding for both goods, and so neither participates in either auction.* ■

3.2.9 Boundaries between regions

It is important to note that, while we have only characterized buyer equilibria for the *interiors* of the regions in $\mathcal{R} \equiv \{1_H 2_{mix}, 2_L 1_{mix}, 1_H 2_L, 1_H 2_H, 1_L 2_L, i_H^{\max H} i_O^{\min H}, i_L^{\max L} i_O^{\min L}, 1_O 2_O\}$, the payoffs to buyers across bordering regions are continuous, and hence any reservation price pair (r^H, r^L) belonging to multiple regions of $\mathcal{R}$ is associated with equilibria described in *each* of those regions. This fact can be shown through simple inspection of the payoffs at the boundaries between different regions in $\mathcal{R}$, and will play a role when we analyze the market equilibrium in the next section.

4 Market Equilibrium: Endogenizing Reservation Prices

In this section, we will endogenize the reservation prices for the auctions by allowing the sellers to announce and commit to reservation prices r^H and r^L before the buyer game is played. To do so, we will first calculate seller payoffs resulting from the buyer equilibria in the eight regions analyzed in section 3.2, and then calculate the equilibrium of the game in which sellers simultaneously announce reservation prices before the buyer game is played. We find that when sellers can commit to announced reservation prices in this manner, there is a unique equilibrium allocation and unique equilibrium payoffs, both of which are equivalent to the case in which sellers simply post prices.

4.1 Seller payoffs resulting from the eight regions of buyer equilibria

We will now calculate the seller payoffs, denoted by π^j, resulting from the buyer equilibria in the eight regions belonging to the set $\mathcal{R} \equiv \{1_H 2_{mix}, 2_L 1_{mix}, 1_H 2_L, 1_H 2_H, 1_L 2_L, i_H^{\max H} i_O^{\min H}, i_L^{\max L} i_O^{\min L}, 1_O 2_O\}$. The calculations are straightforward and mostly mechanical, so we present the payoffs here and relegate the calculations to the Appendix.

Claim 4 *Payoffs for the sellers resulting from the buyer equilibria calculated in propositions 1-8 are as follows:*

region	π^H	π^L
$1_H 2_{mix}$	$\Lambda(r^L, v_1^H, v_2^H, v_1^L, v_2^L)$	$\Upsilon(r^L, r^H, v_1^H, v_2^H, v_2^L)$
$2_L 1_{mix}$	$\Upsilon(r^H, r^L, v_2^L, v_2^L, v_1^H)$	$\Lambda(r^H, v_2^L, v_1^L, v_2^H, v_1^H)$
$1_H 2_L$	r^H	r^L
$1_H 2_H$	$v_{i^{\min H}}^H$	0
$1_L 2_L$	0	$v_{i^{\min L}}^L$
$i_H^{\max H} i_O^{\min H}$	r^H	0
$i_L^{\max L} i_O^{\min L}$	0	r^L
$1_O 2_O$	0	$0,$

where

$$\Lambda(a, b, c, d, e) \equiv \frac{c(b - a + e) + ab - a^2 + e(2a - b - e)}{bc}$$

and

$$\Upsilon(\alpha, \beta, \gamma, \delta, \epsilon) = \frac{\alpha(\gamma - \delta + \epsilon - \alpha)}{\gamma - \beta}.$$

Unlike buyer payoffs, seller payoffs are not continuous across all regions of $\mathcal{R}$, since sellers are in effect bidding each other down in some areas of the reservation price space, hence creating a Bertrand pricing effect with discontinuous payoffs.

Having calculated seller payoffs for the different regions of the reservation price space, it is now straightforward to calculate the seller equilibrium and hence an overall equilibrium of the entire game when sellers can commit to reservation prices. First, we will define a market equilibrium.

Definition 2 *A market equilibrium is a seller strategy profile* (r^{H*}, r^{L*}), *a buyer strategy profile* $\left(\theta_i, F_i^j(\cdot)\right)_{i=1,2}^{j=H,L}$, *and scalars* $\left(\Pi_i^j, \pi^j\right)_{i=1,2}^{j=H,L}$ *such that:*

1. $\left(\theta_i, F_i^j(\cdot)\right)_{i=1,2}^{j=H,L}, \left(\Pi_i^j\right)_{i=1,2}^{j=H,L}$ *is an equilibrium of the buyer game.*
2. $\pi^H = \pi^H(r^{H*}, r^{L*}) = \underset{r^H \in \mathbb{R}_+}{\arg\max}\ \pi^j\left(r^H, r^{L*}\right)$ *and* $\pi^L = \pi^L(r^{H*}, r^{L*}) = \underset{r^L \in \mathbb{R}_+}{\arg\max}\ \pi^L\left(r^{H*}, r^L\right)$.

We can now characterize the unique equilibrium of the market game.

Theorem 2 *The market equilibrium is such that*

$$
\begin{aligned}
r^{H*} &= \min\left\{v_1^H, v_1^H - \left(v_1^L - v_2^L\right)\right\}, \\
r^{L*} &= \min\left\{v_2^L, v_2^L - \left(v_2^H - v_1^H\right)\right\}, \\
\theta_1 &= 1, \\
\theta_2 &= 0, \\
F_1^H(b) &= \begin{cases} 0 & \text{for } b < r^{H*}, \\ 1 & \text{for } b \geq r^{H*}, \end{cases} \\
F_2^L(b) &= \begin{cases} 0 & \text{for } b < r^{L*}, \\ 1 & \text{for } b \geq r^{L*}, \end{cases} \\
\Pi_1 &= \Pi_1^H = \max\{0, v_1^L - v_2^L\}, \\
\Pi_2 &= \Pi_2^L = \max\{0, v_2^H - v_1^H\}, \\
\pi^H &= r^{H*}, \\
\pi^L &= r^{L*}.
\end{aligned}
$$

Proof. *In Appendix.* ■

5 Conclusion

This paper has studied a market in which two buyers with heterogenous but commonly known preferences must choose which one of two different goods–a higher value good and a lower value good–to bid on when the goods are sold through simultaneously held first-price auctions.

We characterized a unique buyer equilibrium for a generic subset of the parameter space when reservation prices are exogenously set to zero. This equilibrium exhibits coordination frictions and results in a potentially inefficient allocation of the goods. The higher value good is sold for sure, though it could be to either buyer and could even be at the reservation price. The lower value good is either not sold or sold exactly at the reservation price.

We then characterized buyer equilibria for all reservation prices. Endogenizing seller behavior by allowing sellers to announce and commit to reservation prices resulted in a unique market equilibrium of the game in which buyers perfectly coordinate, the efficient allocation is achieved, and both allocation and payoffs are equivalent to that of the price-posting case.

The contrast in results between when sellers can commit to reservation prices and when buyers can behave knowing that sellers' "true" reservation prices of zero are in effect is notable. We can obtain from this result a better understanding of the importance of which side of a decentralized market is effectively responsible for price formation when there is two-sided heterogeneity.

When the same side that *searches* (here, the buyer side) is responsible for price formation, coordination frictions are still present because buyers have incentive to compete over the higher value good and hence could both bid on it. However, when the *passive* side of the market (here, the seller side) is in effect responsible (here, by setting *binding* reservation prices), the two-sided heterogeneity results in perfect coordination by the buyers in choosing their goods. The buyer with higher marginal valuation for the higher value good receives it at reservation price, while the other buyer receives the low value good at its reservation price.

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6 Appendix

Proof of Lemma 1. Suppose first that $v_1^j = v_2^j$. Then this is an exact analog to Bertrand competition between identical firms, in which the only equilibrium is that both buyers (firms) choose the bid (price) equal to their value (cost) with probability one. This equilibrium satisfies all of the conditions described in this lemma.

The rest of the proof will consider the case in which $v_1^j > v_2^j$.

Fact 1 It cannot be that F_i^j has a discontinuity at any bid $\widehat{b} \in [0, v_2^j)$ unless $\Pi_i^j = 0$, for either $i = 1, 2$.

Proof of Fact 1 Suppose F_1^j has a discontinuity at some $\widehat{b} \in [0, v_2^j)$. If $\widehat{b} \in \mathcal{S}_2^j$, then buyer 2 must not be able to achieve a higher payoff than he achieves with $\widehat{b}$, unless $\widehat{b}$ is the only point in its neighborhood which achieves a lower payoff. Due to buyer 1 playing $\widehat{b}$ with positive probability, however, there is a positive probability of a tie at $\widehat{b}$ when buyer 2 bids $\widehat{b}$, and a positive probability of buyer 2 losing to a bid of $\widehat{b}$ when buyer 2 bids slightly below $\widehat{b}$. So buyer 2 can bid slightly above $\widehat{b}$ and achieve a higher payoff than he does at a positive measure of points including $\widehat{b}$ and the points slightly below it. (The infinitesimal increase in bid is more than offset for buyer 2 by the discontinuous increase in winning probability.) So it cannot be that $\widehat{b} \in \mathcal{S}_2^j$. If $\widehat{b} \notin \mathcal{S}_2^j$, however, then it is suboptimal for buyer 1 to bid $\widehat{b}$ (unless $\widehat{b} < \min \mathcal{S}_2^j$, but then $\Pi_1^j = 0$), since bidding slightly less does not decrease his probability of winning since buyer 2 is not bidding in the neighborhood just below $\widehat{b}$. The same logic applies for discontinuities in F_2^j.

Fact 2 If both F_1^j and F_2^j are continuous, then it must hold that the supports $\mathcal{S}_i^j$ are convex for both $i = 1, 2$.

Proof of Fact 2 Suppose $\mathcal{S}_1^j$ is not convex. That is, suppose F_1^j is constant on some region $[b', b'']$ in the convex hull of it support. If $v_2^j > b'$, then buyer 2 is strictly better off bidding b' than bidding any value on the interval $(b', b'']$, since his probability of winning good j is constant over the entire interval. This implies that F_2^j is also constant on the same region $[b', b'']$, making it suboptimal for buyer 1 to have b'' and the neighborhood just above b'' in $\mathcal{S}_1^j$. If $v_2^j \leq b'$, then it must be that $\overline{b_2^j} \leq b'$, since any bids in the interval $[b', b'')$ would then yield a negative expected payoff. It is then suboptimal for buyer 1 to bid any $b \geq b''$, since bids in the interval (b', b'') also yield probability one of winning good j but at a lower cost. The same logic applies if we suppose that $\mathcal{S}_2^j$ is not convex.

Fact 3 It must hold that $\overline{b_1^j} = \overline{b_2^j}$.

Proof of Fact 3 If $\overline{b_1^j} > \overline{b_2^j}$, then buyer 1 wins the auction for all bids in the interval $(\overline{b_2^j}, \overline{b_1^j}]$, and hence all bids in this interval are suboptimal since he can always bid slightly lower and still win with probability one. The same logic applies for buyer 2 if $\overline{b_1^j} < \overline{b_2^j}$.

Fact 4 Both F_1^j and F_2^j are continuous if and only if $\mathcal{S}_1^j = \mathcal{S}_2^j (= \mathcal{S}^j)$.

Proof of Fact 4 Suppose that both F_1^j and F_2^j are continuous. From fact 2 we have that $\mathcal{S}_1^j$ and $\mathcal{S}_2^j$ are both convex, so $\mathcal{S}_1^j \neq \mathcal{S}_2^j$ implies that either $\overline{b_1^j} \neq \overline{b_2^j}$ or $\min \mathcal{S}_1^j \neq \min \mathcal{S}_2^j$. Also, we can establish that $\overline{b_2^j} < v_2^j$. (If $\overline{b_2^j} \geq v_2^j$, then buyer 2 would have bids in his support at which he has a positive probability of winning and receiving a negative payoff, which is not possible in equilibrium, unless it were the case that $\overline{b_2^j} \le \min\, \mathcal{S}_1^j$, which is also impossible since we know F_1^j is continuous, and yet all bids strictly above $\overline{b_2^j}$ would yield buyer 1 a strictly lower payoff than at $\overline{b_2^j}$ if $\overline{b_2^j} \le \min\, \mathcal{S}_1^j$.) We know $\overline{b_1^j} = \overline{b_2^j}$ from fact 3. Suppose that $\min\, \mathcal{S}_1^j \neq \min\, \mathcal{S}_2^j$. But if $\min\, \mathcal{S}_1^j < \min\, \mathcal{S}_2^j$, then since $\Pi_1^j(b) = 0$ for all $b < \min\, \mathcal{S}_2^j$ and some such bids are in $\mathcal{S}_1^j$, it cannot hold that buyer 1 is indifferent between all bids in his support, since $\Pi_1^j(b) > 0$ for any bid $b > \min\, \mathcal{S}_2^j$. The same logic applies for if $\min\, \mathcal{S}_1^j > \min\, \mathcal{S}_2^j$. Now suppose that $\mathcal{S}_1^j = \mathcal{S}_2^j (= \mathcal{S}^j)$ and that either F_1^j or F_2^j has a discontinuity. Because $\mathcal{S}_1^j = \mathcal{S}_2^j$ both buyers have a positive probability of winning the auction at all bids $b \in \mathcal{S}^j \setminus \min \mathcal{S}^j$, and hence a positive payoff, unless $\mathcal{S}^j = \left\{ v_2^j \right\}$, which is impossible since buyer 1 would deviate to bid slightly more and win for sure. Fact 1 then tells us that F_i^j cannot have a discontinuity at any bid $\widehat{b} \in [0, v_2^j)$ for either $i = 1, 2$. Suppose the discontinuity is at some $b' > v_2^j$. Then, regardless of whether the discontinuity is in F_1^j or F_2^j, buyer 2 must be getting a negative payoff for some $b \in (v_2^j, \overline{b^j}] \cap \mathcal{S}^j \neq \varnothing$, which cannot occur in equilibrium. But a discontinuity at v_2^j implies that either $\mathcal{S}^j = \left\{ v_2^j \right\}$, which is impossible as stated above, or that buyer 2 is getting a limit payoff of $\Pi_2^j = \lim_{b \nearrow v_2^j} (v_2^j - b) F_1^j(b) = 0$ when bidding close to v_2^j but $\Pi_2^j(b) = (v_2^j - b) F_1^j(b) > 0$ for $b < v_2^j$, which violates his indifference condition between almost all bids $b \in \mathcal{S}^j$.

Fact 5 There are no equilibria in which both F_1^j and F_2^j are continuous (and hence, by corollary, no equilibria in which $\mathcal{S}_1^j = \mathcal{S}_2^j (= \mathcal{S}^j)$).

Proof of Fact 5 Using facts 2 and 4, we can use a limit to argument to state that

$$\begin{aligned} \Pi_1^j &= \lim_{b \searrow \min \mathcal{S}^j} \Pi_1^j(b) \\ &= \lim_{b \searrow \min \mathcal{S}^j} (v_1^j - b) \underbrace{F_2^j(b)}_{\to 0 \text{ as } b \searrow \min \mathcal{S}^j} \\ &= 0, \end{aligned}$$

which cannot hold in equilibrium since we know that $\Pi_1^j(b) > 0$ for any bid $b > \min\, \mathcal{S}^j$. Fact 1 then gives us that discontinuities can only arise at or above v_2^j.

We now have that $\overline{b_1^j} = \overline{b_2^j}$, $\mathcal{S}_1^j \neq \mathcal{S}_2^j$, at least one of F_1^j and F_2^j have a discontinuity, and that this discontinuity must be at some $b > v_2^j$ unless the buyer with the discontinuity in his distribution gets a payoff of zero. Next, we will show that the only overlap between the two supports in equilibrium can be at the maximum.

Fact 6 It must hold that $\mathcal{S}_1^j \cap \mathcal{S}_2^j = \left\{ \overline{b^j} \right\}$.

Proof of Fact 6 We know from fact 3 that $\overline{b^j} \in \mathcal{S}_1^j \cap \mathcal{S}_2^j$. Now suppose that some $b' \in \mathcal{S}_1^j \cap \mathcal{S}_2^j$ such that $b' < \overline{b^j}$. Because this implies $F_1^j(\overline{b^j}) > 0$, we have that $\Pi_2^j(\overline{b^j}) = (v_2^j - \overline{b^j}) F_2^j(\overline{b^j})$ has the same sign as $v_2^j - \overline{b^j}$. If $v_2^j - \overline{b^j} < 0$, then buyer 2 is getting a negative payoff for bids close to $\overline{b^j}$. If $v_2^j - \overline{b^j} = 0$, then buyer 2 is getting a limit payoff of zero for bids close to $\overline{b^j}$ but a positive payoff below $\overline{b^j}$, which cannot occur. Finally, if $v_2^j - \overline{b^j} > 0$, then buyer 2 gets a positive payoff from the auction and hence must be doing so at bids near the bottom of the support as well, implying that either $\min\, \mathcal{S}_1^j < \min\, \mathcal{S}_2^j$ or that $\min\, \mathcal{S}_1^j = \min\, \mathcal{S}_2^j$ and $F_1^j(\min\, \mathcal{S}_1^j) > 0$. If $\min\, \mathcal{S}_1^j < \min\, \mathcal{S}_2^j$, then $\Pi_1^j = \lim_{b \to \min \mathcal{S}_1^j} \Pi_1^j(b) = 0$, but we know that $\Pi_1^j(\overline{b^j}) = (v_1^j - \overline{b^j}) F_2^j(\overline{b^j}) > 0$, since $b' \in \mathcal{S}_1^j \cap \mathcal{S}_2^j$ and $b' < \overline{b^j}$ implies $F_2^j(\overline{b^j}) > 0$ and we have that $v_1^j > v_2^j > \overline{b^j}$. If min $\mathcal{S}_1^j$ = min $\mathcal{S}_2^j$ and $F_1^j(\min \mathcal{S}_1^j) > 0$, we again get $\Pi_1^j = \lim_{b \to \min \mathcal{S}_1^j} \Pi_1^j(b) = 0$, unless $F_2^j(\min \mathcal{S}_1^j) > 0$ as well, but both buyers cannot have a mass point at min $\mathcal{S}_1^j$, since there is then a positive probability of a tie and bidding slightly higher will lead to a discontinuous increase in payoff.

Fact 7 It must hold that $\mathcal{S}_1^j = \left\{\overline{b^j}\right\}$ and $\lim_{b \nearrow \overline{b^j}} F_2^j(b) = 1$.

Proof of Fact 7 It is immediate from fact 6 that either $\mathcal{S}_1^j = \left\{\overline{b^j}\right\}$ or $\mathcal{S}_2^j = \left\{\overline{b^j}\right\}$ and that the payoff for the other buyer must be zero. If $\mathcal{S}_2^j = \left\{\overline{b^j}\right\}$, then $\Pi_1^j = 0$ and $\Pi_2^j = v_2^j - \overline{b^j} \geq 0$, but because $v_1^j > v_2^j$ buyer 1 could profitably deviate to bidding some $b \in (\overline{b^j}, v_1^j)$. So it must be that $\mathcal{S}_1^j = \left\{\overline{b^j}\right\}$, and then $\lim_{b \nearrow \overline{b^j}} F_2^j(b) = 1$ must hold in order to avoid a positive probability of a tie at $\overline{b^j}$, which would give buyer 1 incentive to deviate to a slightly higher bid.

Now, it must hold that $\overline{b^j} \geq v_2^j$, or buyer 2 could deviate to some bid $b \in (\overline{b^j}, v_2^j)$ and get a positive payoff. Buyer 1 is receiving a payoff of $\Pi_1^j = v_1^j - \overline{b^j}$, and so it must also hold that $\overline{b^j} < v_1^j$, since otherwise $\Pi_1^j \leq 0$ and buyer 1 could obtain a positive payoff by bidding slightly less than v_1^j. Also, it is immediate that $\Pi_2^j = 0$. We are left only to show that it is necessary for $F_2^j(b) \leq \frac{v_1^j - \overline{b^j}}{v_1^j - b}$ for all $b \leq \overline{b^j}$, but this follows directly from buyer 1's incentive not to deviate to lower bids than $\overline{b^j}$, which can be written as $\Pi_1^j = v_1^j - \overline{b^j} \geq (v_1^j - b)F_2^j(b)$ for all $b \leq \overline{b^j}$. ■

Proof of Lemma 2. Without loss of generality, let $v_1^j \geq v_2^j$. Suppose now that F_1^j has a discontinuity at some $\widehat{b} > 0$, so buyer 1 is bidding $\widehat{b}$ with positive probability. We will break the argument into two cases: (i) $\widehat{b} < v_2^j$ and (ii) $\widehat{b} \geq v_2^j$. Case (i):$\widehat{b} < v_2^j$. First suppose there exists some $\eta > 0$ such that $\left[\widehat{b} - \eta, \widehat{b}\right] \in \mathcal{S}_2^j$. Then for some interval $(\widehat{b} - \eta', \widehat{b}]$ it must hold that $\Pi_2^j(b) < \Pi_2^j(\widehat{b} + \varepsilon)$ for all $b \in (\widehat{b} - \eta', \widehat{b})$ for η' and ε sufficiently small. Now suppose there exists no $\eta > 0$ such that $\left[\widehat{b} - \eta, \widehat{b}\right] \in \mathcal{S}_2^j$. Then it is immediate that there exists a $\eta'' > 0$ such that $\Pi_1^j(b) > \Pi_1^j(\widehat{b})$ for all $b \in (\widehat{b} - \eta'', \widehat{b})$. Case (ii):$\widehat{b} \geq v_2^j$. If buyer 1 places probability one on some $\widehat{b} \geq v_2^j$, then $\Pi_2^j \leq 0$ so long as buyer 1 participates in auction j. But then buyer 2's best response is to bid zero on good j, making buyer 1's bid of $\widehat{b}$ suboptimal. So it must hold that $\min \mathcal{S}_1^j < \widehat{b}$, allowing for $\Pi_2^j > 0$ even when buyer 1 participates in auction j. Now suppose there exists some $\eta > 0$ such that $\left[\widehat{b} - \eta, \widehat{b}\right] \in \mathcal{S}_2^j$. If $\widehat{b} > v_2^j$ then $\Pi_2^j(b) < 0$ for all $b \in \left(v_2^j, \widehat{b}\right) \cap \mathcal{S}_2^j \neq \varnothing$, so $\widehat{b} = v_2^j$ must hold. But then $\lim_{b \nearrow v_2^j} \Pi_2^j(b) = 0$, violating buyer 2's indifference condition that $\Pi_2^j(b) = \Pi_2^j$ for almost all $b \in \mathcal{S}_2^j$. ■

Proof of Lemma 3. Suppose $\mathcal{S}_1^j$ is not convex. That is, suppose F_1^j is constant on some region $[b', b'']$ in the convex hull of it support. If $v_2^j > b'$, then buyer 2 is strictly better off bidding b' than bidding any value on the interval $(b', b'']$, since his probability of winning good j is constant over the entire interval. This implies that F_2^j is also constant on the same region $[b', b'']$, making it suboptimal for buyer 1 to have b'' and the neighborhood just above b'' in $\mathcal{S}_1^j$. If $v_2^j \leq b'$, then it must be that $\overline{b_2^j} \leq b'$, since any bids in the interval $[b', b'')$ would then yield a negative expected payoff. It is then suboptimal for buyer 1 to bid any $b \geq b''$, since bids in the interval (b', b'') also yield probability one of winning good j but at a lower cost. ■

Proof of Lemma 4. Since both buyers are placing less than probability one on bidding for the other good, both buyers can guarantee a positive expected payoff by bidding less than his value in the other auction. Therefore, it must hold that $\overline{\underline{b_i^j}} < v_i^j$ for both $i = 1, 2$. Furthermore, it must hold that $\overline{b_i^j} > 0$ for both $i = 1, 2$. (If it were the case that $\overline{b_1^j} = 0$, then buyer 2 would always have the profitable deviation of choosing a smaller $\varepsilon > 0$ to bid on good j.) From lemma 2, we know that F_2^j cannot have any mass points at positive bids, *.and* so if buyer 1 bids $\overline{b_2^j}$, he wins good j with probability one. Bidding any higher is therefore suboptimal. The same is true for buyer 2 when he bids $\overline{b_1^j}$. Therefore, it cannot be that $\overline{b_1^j} > \overline{b_2^j}$ or $\overline{b_2^j} > \overline{b_1^j}$, so it must hold that $\overline{b_1^j} = \overline{b_2^j}$ ($\equiv \overline{b^j}$), which proves part (i). Now consider the low end of the bidding distributions F_1^j and F_2^j. Suppose $\min \mathcal{S}_1^j < \min \mathcal{S}_2^j$. Because buyer 1 is certain to lose the auction when bidding $\min \mathcal{S}_1^j$ whenever buyer 2 participates, it must be that $\min \mathcal{S}_1^j = 0$. (Bidding any higher than zero but lower than $\min \mathcal{S}_2^j$ would be suboptimal for buyer 1 since he would still never win the auction for j when buyer 2 participates, and will have to pay a positive bid when buyer 2 does not participate.) But if $\min \mathcal{S}_1^j = 0$ and bidding anything slightly higher is suboptimal, then $\mathcal{S}_1^j$ must not be convex, which contradicts the result of lemma 3. ■

Proof of Lemma 5. From lemma 4, we have that $\overline{b_1^j} = \overline{b_2^j} (\equiv \overline{b^j})$, and so $\overline{b^j} \in \mathcal{S}_i^j$ for both $i = 1, 2$. From the convexity of $\mathcal{S}_i^j$ (from lemma 3) we have that for all sufficiently small $\varepsilon > 0$, and letting $\theta_{i'}^j$ denote the probability the other buyer places on the auction for good j,

$$\begin{aligned}
\Pi_i^j &= \lim_{\varepsilon \to 0} \Pi_i^j(\overline{b^j} - \varepsilon) \\
&= \lim_{\varepsilon \to 0} (v_i^j - (\overline{b^j} - \varepsilon))(1 - \theta_{i'}^j + \theta_{i'}^j \underbrace{F_2^H(\overline{b^H} - \varepsilon)}_{\to 1 \text{ as } \varepsilon \to 0}) \\
&= v_i^j - \overline{b^j}.
\end{aligned}$$

■

Proof of Lemma 6. Suppose $F_1^j(0) > 0$ and $F_2^j(0) > 0$. We know from lemmas **??** and **??** that both players are bidding on good j with positive probability, and therefore have their expected payoffs affected by the bidding for good j. But if both players place positive probability on bidding zero for good j, either player could profitably deviate to bidding some arbitrarily small $\varepsilon > 0$ and avoiding the positive probability of a tie. ■

Proof of Claim 4. First consider region $1_H 2_{mix}$ and let $G^H(p)$ be the equilibrium price distribution for good H. With probability $1 - \theta_2$, buyer 2 will not bid on good H and the price distribution G^H will simply be identical to F_1^H. With probability θ_2, however, both buyers will bid for good H and the price distribution will be given by $G^H(p) = \Pr\{\max\{b_1^H, b_2^H\} \le p\} = \Pr\left\{b_1^H \le p\right\} \cdot \Pr\left\{b_1^L \le p\right\} = F_1^H(p) F_2^H(p)$. Overall, we therefore have that

$$\begin{aligned}
G^H(p) &= F_1^H(p)\left[(1 - \theta_2) + \theta_2 F_2^H\right] \\
&= \left(\frac{v_2^L - r^L}{v_2^H - p}\right)\left[\frac{v_1^H - v_2^H + v_2^L - r^L}{v_1^H - p}\right].
\end{aligned}$$

Let $g^H(p) = \frac{dG^H(p)}{dp}$. Seller H then receives an expected payoff of

$$\begin{aligned}
\pi^H &= \int_0^{\overline{p^H}} p dG^H(p) \\
&= \int_0^{\overline{p^H}} p g^H(p) dp \\
&= \int_0^{v_2^H - v_2^L + r^L} p \left[\frac{\left(v_2^L - r^L\right)\left(v_1^H - v_2^H + v_2^L - r^L\right)\left(v_1^H - 2p + v_2^H\right)}{\left(v_2^H - p\right)^2 \left(v_1^H - p\right)^2}\right] dp \\
&= \Lambda(r^L, v_1^H, v_2^H, v_1^L, v_2^L),
\end{aligned}$$

where $\Lambda(a,b,c,d,e) \equiv \frac{c(b-a+e)+ab-a^2+e(2a-b-e)}{bc}$. Meanwhile, good L is sold with probability $1-\theta_2$ for a price of r^L, and is not sold otherwise. Seller L hence receives an expected payoff of

$$\begin{aligned} \pi^L &= r^L(1-\theta_2) \\ &= \Upsilon(r^L, r^H, v_1^H, v_2^H, v_2^L), \end{aligned}$$

where $\Upsilon(\alpha,\beta,\gamma,\delta,\epsilon) = \frac{\alpha(\gamma-\delta+\epsilon-\alpha)}{\gamma-\beta}$. The logic for region 2_L1_{mix} is exactly the same, with the names of the goods and buyers reversed. Seller payoff for the other six regions are immediate. ■

Proof of Theorem 2. We will first derive sellers' best response correspondences in their announcement of reservation prices. Note that $\Lambda(r^L, v_1^H, v_2^H, v_1^L, v_2^L)$ is independent of r^H and $\Lambda(r^H, v_2^L, v_1^L, v_2^H, v_1^H)$ is independent of r^L, whereas $r^H_{\max} \equiv \arg\max_{r^H} \Upsilon(r^H, r^L, v_2^L, v_2^L, v_1^H) = \frac{v_2^L - v_1^L + v_1^H}{2}$ and $r^L_{\max} \equiv \arg\max_{r^L} \Upsilon(r^L, r^H, v_1^H, v_2^H, v_2^L) = \frac{v_1^H - v_2^H + v_2^L}{2}$. Abusing notation by allowing $\varepsilon > 0$ be some arbitrarily small amount to symbolize undercutting, the best response correspondence (which happens to be a function) for seller H is then given by

$$BR_H(r^L) = \begin{cases} r^L + v_1^H - v_1^L & \text{for } r^L \le \min\{v_1^L, v_2^L\}, \\ v_1^H & \text{for } v_1^L \le r^L \le v_2^L - v_2^H + v_1^H, \\ r^L + v_1^H - v_1^L - \varepsilon & \text{for } v_2^L \le r^L \le v_1^L, \\ r^L + v_2^H - v_2^L - \varepsilon & \text{for } v_2^L - v_2^H + v_1^H \le r^L \le v_2^L, \\ \max\left\{v_1^H, v_2^H\right\} & \text{for } r^L > \max\left\{v_1^L, v_2^L\right\}. \end{cases}$$

The best response correspondence for seller L is given by

$$BR_L(r^H) = \begin{cases} \min\{\max\{r^L_{\max}, r^H - v_2^H + v_2^L\}, v_2^L\} & \text{for } r^H \le \min\{v_1^H, v_1^H - v_1^L + v_2^L\}, \\ r^H - v_1^H + v_1^L - \varepsilon & \text{for } v_1^H - v_1^L + v_2^L \le r^H \le v_1^H, \\ r^H - v_2^H + v_2^L - \varepsilon & \text{for } v_1^H \le r^H \le v_2^H, \\ \max\{v_1^L, v_2^L\} & \text{for } r^H > \max\{v_1^H, v_2^H\}. \end{cases}$$

The best response correspondences intersect only at the point $(r^{H*}, r^{L*}) = (\min\left\{v_1^H, v_1^H - \left(v_1^L - v_2^L\right)\right\}, \min\left\{v_2^L, v_2^L - \left(v_2^H - v\right.\right.$ Mixed strategy equilibria can then be ruled out using iterative elimination of strictly dominated strategies and standard results regarding putting positive probability on pure strategies that are never a best reply. ■

6.1 Equilibrium when $v_1^H - v_1^L = v_2^H - v_2^L$

Having ruled out the existence of equilibria in all cases except for when $v_1^H - v_1^L = v_2^H - v_2^L$, we will now solve for the equilibria given this restriction as a parametric condition. To simplify notation, let $d \equiv v_1^H - v_1^L = v_2^H - v_2^L$ denote the common difference in value that both buyers place between the two goods. We can then consider the model as having just three parameters: v_1^H, v_2^H, and d.

We will begin by analyzing payoffs for buyer 1 at the lower end of his bidding distribution for good H. We know from lemma 6 that both buyers cannot bid zero with positive probability on the same good. Let $p_1^H \equiv F_1^H(0) \geq 0$ and assume (without loss of generality) that $F_2^H(0) = 0$. When buyer 1 bids zero on good H, he only wins if buyer 2 chooses not to participate in the auction for H, and so buyer 1's payoff is then $\Pi_1^H(0) = v_1^H(1-\theta_2)$. Since $\{0, \overline{b^H}\} \in \mathcal{H}_1$, it must then hold that $\Pi_1^H = \Pi_1^H(\overline{b^H}) = \Pi_1^H(0)$, or

$$\Pi_1^H = v_1^H - \overline{b^H} = v_1^H(1-\theta_2). \tag{1}$$

Furthermore, it must hold that for any $b \in \mathcal{H}_1$,

$$\Pi_1^H = (v_1^H - b)\left[(1-\theta_2) + \theta_2 F_2^H(b)\right]. \tag{2}$$

For buyer 2, we know that $\Pi_2^H = v_2^H - \overline{b^H}$. However, when $p_1^H > 0$, we must now consider the limit payoff for buyer 2 when bidding arbitrarily close to zero. We have that

$$\begin{aligned} \lim_{\varepsilon \to 0} \Pi_2^H(\varepsilon) &= \lim_{\varepsilon \to 0} \left(v_2^H - \varepsilon\right)[(1-\theta_1) + \theta_1 \underbrace{F_1^H(\varepsilon)}_{\to p_1^H \text{ as } \varepsilon \to 0}] \\ &= v_2^H[(1-\theta_1) + \theta_1 p_1^H], \end{aligned}$$

and equating $\Pi_2^H = \lim_{\varepsilon \to 0} \Pi_2^H(\varepsilon)$ we get

$$\Pi_2^H = v_2^H - \overline{b^H} = v_2^H[(1-\theta_1) + \theta_1 p_1^H]. \tag{3}$$

Furthermore, it must hold that for any $b \in \mathcal{H}_2$,

$$\Pi_2^H = (v_2^H - b)\left[(1-\theta_1) + \theta_1 F_1^H(b)\right]. \tag{4}$$

We have two cases to consider as we move on to the analysis of the auction for good L. The first is the case in which buyer 2 (the buyer other than the one who possibly bid zero with positive probability on good H), possibly bids zero with positive probability on good L. The other is the case in which the same buyer (buyer 1) who possibly bid zero with positive probability on good H also possibly bids zero with positive probability on good L. We say "possibly" to emphasize that it is not a necessary condition that either buyer places positive probability on zero in a given auction. In fact, the only restriction, obtained in lemma 6, is that both buyers cannot place positive probability on zero in the same auction.

Case 1: $p_2^L \equiv F_2^L(0) \geq 0,\ F_1^L(0) = 0.$ Using similar arguments to those used in the analysis of the good H auction, we have that

$$\Pi_2^L = v_2^L - \overline{b^L} = v_2^L \theta_1, \tag{5}$$

that it must hold that for any $b \in \mathcal{L}_2$ that

$$\Pi_2^L = (v_2^L - b)\left[\theta_1 + (1-\theta_1) F_1^L(b)\right], \tag{6}$$

and also that

$$\Pi_1^L = v_1^L - \overline{b^L} = v_1^L[\theta_2 + (1-\theta_2) p_2^L]. \tag{7}$$

and that it must hold that for any $b \in \mathcal{L}_1$ that

$$\Pi_1^L = (v_1^L - b)\left[\theta_2 + (1-\theta_2) F_2^L(b)\right]. \tag{8}$$

Setting $\Pi_1^H = \Pi_1^L$ yields

$$v_1^H - \overline{b^H} = v_1^H(1-\theta_2) = v_1^L - \overline{b^L} = v_1^L[\theta_2 + (1-\theta_2)p_2^L],$$

while setting $\Pi_2^H = \Pi_2^L$ yields

$$v_2^H - \overline{b^H} = v_2^H[(1-\theta_1) + \theta_1 p_1^H] = v_2^L - \overline{b^L} = v_2^L\theta_1.$$

We have a system of six unknowns $(\overline{b^H}, \overline{b^L}, \theta_1, \theta_2, p_1^H, p_2^L)$ and only five equations:

$$\begin{aligned}
\overline{b^H} - \overline{b^L} &= d, \\
v_2^H - \overline{b^H} &= v_2^H[(1-\theta_1) + \theta_1 p_1^H], \\
v_1^L - \overline{b^L} &= v_1^L[\theta_2 + (1-\theta_2)p_2^L], \\
v_1^H - \overline{b^H} &= v_1^H(1-\theta_2), \\
v_2^L - \overline{b^L} &= v_2^L\theta_1.
\end{aligned}$$

It turns out that for any $\overline{b^L} \in (0, \min_{i\in\{1,2\}} \frac{(v_i^L)^2}{v_i^H + v_i^L}]$, the following is an equilibrium:

$$F_1^H(b) = \begin{cases} \frac{v_2^L}{v_2^H - b} - \frac{\overline{b^L}}{v_2^L - \overline{b^L}} & \text{for } b \in \left[0, \overline{b^L} + d\right], \\ 1 & \text{for } b > \overline{b^L} + d, \end{cases}$$

$$F_1^L(b) = \begin{cases} \frac{(v_2^L - \overline{b^L})b}{(v_2^L - b)\overline{b^L}} & \text{for } b \in \left[0, \overline{b^L}\right], \\ 1 & \text{for } b > \overline{b^L}, \end{cases}$$

$$F_2^H(b) = \begin{cases} \frac{(v_1^L - \overline{b^L})b}{(v_2^L - b)(\overline{b^L} + d)} & \text{for } b \in \left[0, \overline{b^L} + d\right], \\ 1 & \text{for } b > \overline{b^L} + d, \end{cases}$$

$$F_2^L(b) = \begin{cases} \frac{v_1^H}{v_1^L - b} - \frac{\overline{b^L} + d}{v_1^L - \overline{b^L}} & \text{for } b \in \left[0, \overline{b^L}\right], \\ 1 & \text{for } b > \overline{b^L}, \end{cases}$$

$$\theta_1 = \frac{v_2^L - \overline{b^L}}{v_2^L},$$

$$\theta_2 = \frac{\overline{b^L} + d}{v_1^H}.$$

Case 2: $p_1^L \equiv F_1^L(0) \geq 0,\ F_2^L(0) = 0.$ Using similar arguments to those used in the analysis of the good H auction, we have that

$$\Pi_2^L = v_2^L - \overline{b^L} = v_2^L\left[\theta_1 + (1-\theta_1)\, p_1^L\right], \tag{9}$$

that it must hold that for any $b \in \mathcal{L}_2$ that

$$\Pi_2^L = (v_2^L - b)\left[\theta_1 + (1-\theta_1)F_1^L(b)\right], \tag{10}$$

and also that

$$\Pi_1^L = v_1^L - \overline{b^L} = v_1^L \theta_2. \tag{11}$$

and that it must hold that for any $b \in \mathcal{L}_1$ that

$$\Pi_1^L = (v_1^L - b)\left[\theta_2 + (1-\theta_2)F_2^L(b)\right]. \tag{12}$$

Setting $\Pi_1^H = \Pi_1^L$ yields

$$v_1^H - \overline{b^H} = v_1^H(1-\theta_2) = v_1^L - \overline{b^L} = v_1^L\theta_2,$$

while setting $\Pi_2^H = \Pi_2^L$ yields

$$v_2^H - \overline{b^H} = v_2^H[(1-\theta_1) + \theta_1 p_1^H] = v_2^L - \overline{b^L} = v_2^L\left[\theta_1 + (1-\theta_1)\,p_1^L\right].$$

We have a system of six unknowns $(\overline{b^H}, \overline{b^L}, \theta_1, \theta_2, p_1^H, p_1^L)$ and only five equations:

$$\begin{aligned}
\overline{b^H} - \overline{b^L} &= d, \\
v_2^H - \overline{b^H} &= v_2^H[(1-\theta_1) + \theta_1 p_1^H], \\
v_2^L - \overline{b^L} &= v_2^L[\theta_1 + (1-\theta_1)p_1^L], \\
v_1^H - \overline{b^H} &= v_1^H(1-\theta_2), \\
v_1^L - \overline{b^L} &= v_1^L\theta_2.
\end{aligned}$$

It turns out that it must hold that $v_1^j < v_2^j$ must hold for both $j = H, L$ for this equilibrium to exist. That is, the buyer who values a given good j lower than the other buyer must be the one who bids zero with positive probability on both goods. Furthermore, we have that for any $\theta_1 \in \left[\frac{\left(v_1^H\right)^2}{v_2^H\left(v_1^H+v_1^L\right)}, 1 - \frac{\left(v_1^L\right)^2}{v_2^L\left(v_1^H+v_1^L\right)}\right]$, the following is an equilibrium:

$$F_1^H(b) = \begin{cases} \frac{v_2^H - \frac{\left(v_1^H\right)^2}{v_1^H+v_1^L}}{\theta_1(v_2^H-b)} - \frac{1-\theta_1}{\theta_1} & \text{for } b \in \left[0, \frac{\left(v_1^H\right)^2}{v_1^H+v_1^L}\right], \\ 1 & \text{for } b > \frac{\left(v_1^H\right)^2}{v_1^H+v_1^L}, \end{cases}$$

$$F_1^L(b) = \begin{cases} \frac{v_2^L - \frac{\left(v_1^L\right)^2}{v_1^H+v_1^L}}{(1-\theta_1)(v_2^L-b)} - \frac{\theta_1}{1-\theta_1} & \text{for } b \in \left[0, \frac{\left(v_1^L\right)^2}{v_1^H+v_1^L}\right], \\ 1 & \text{for } b > \frac{\left(v_1^L\right)^2}{v_1^H+v_1^L}, \end{cases}$$

$$F_2^H(b) = \begin{cases} \frac{v_1^L}{v_1^H}\left(\frac{b}{v_1^H-b}\right) & \text{for } b \in \left[0, \frac{\left(v_1^H\right)^2}{v_1^H+v_1^L}\right], \\ 1 & \text{for } b > \frac{\left(v_1^H\right)^2}{v_1^H+v_1^L}, \end{cases}$$

$$F_2^L(b) = \begin{cases} \frac{v_1^H}{v_1^L}\left(\frac{b}{v_1^L-b}\right) & \text{for } b \in \left[0, \frac{\left(v_1^L\right)^2}{v_1^H+v_1^L}\right], \\ 1 & \text{for } b > \frac{\left(v_1^L\right)^2}{v_1^H+v_1^L}, \end{cases}$$

$$\theta_2 = \frac{v_1^H}{v_1^H + v_1^L}.$$

Figure 1: Both buyers cannot be indifferent between goods generically

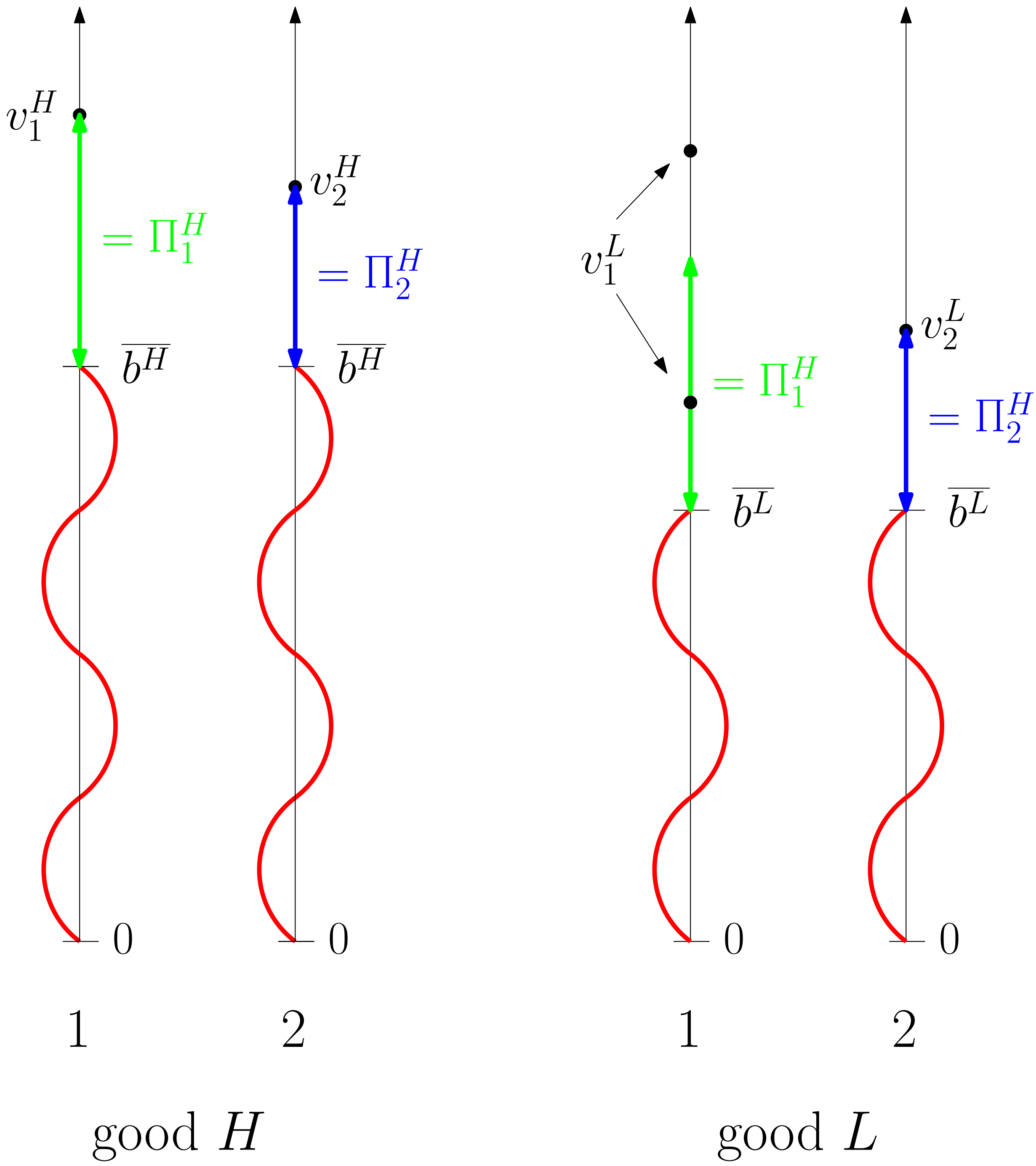
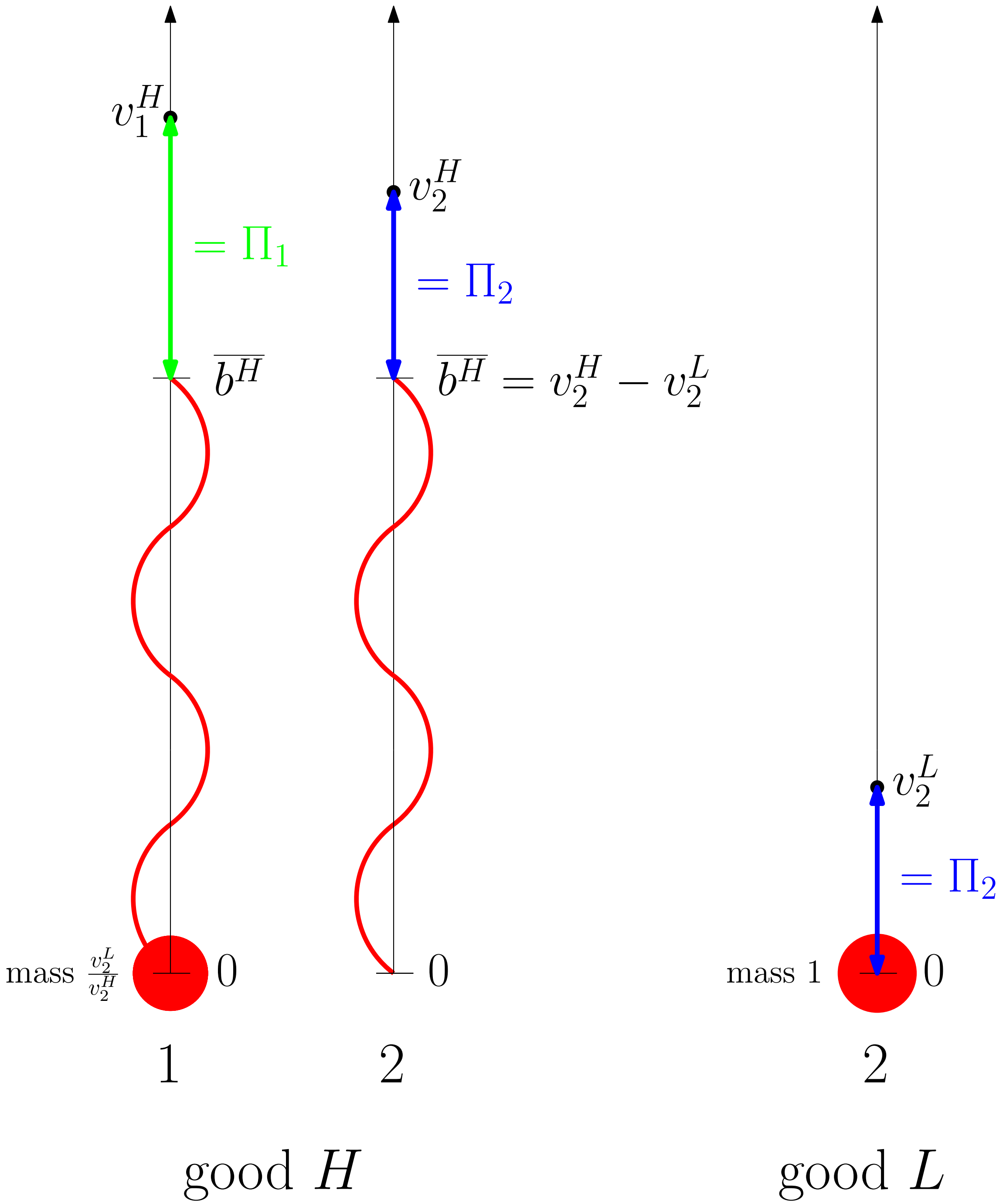


Figure 2: The unique generic buyer equilibrium



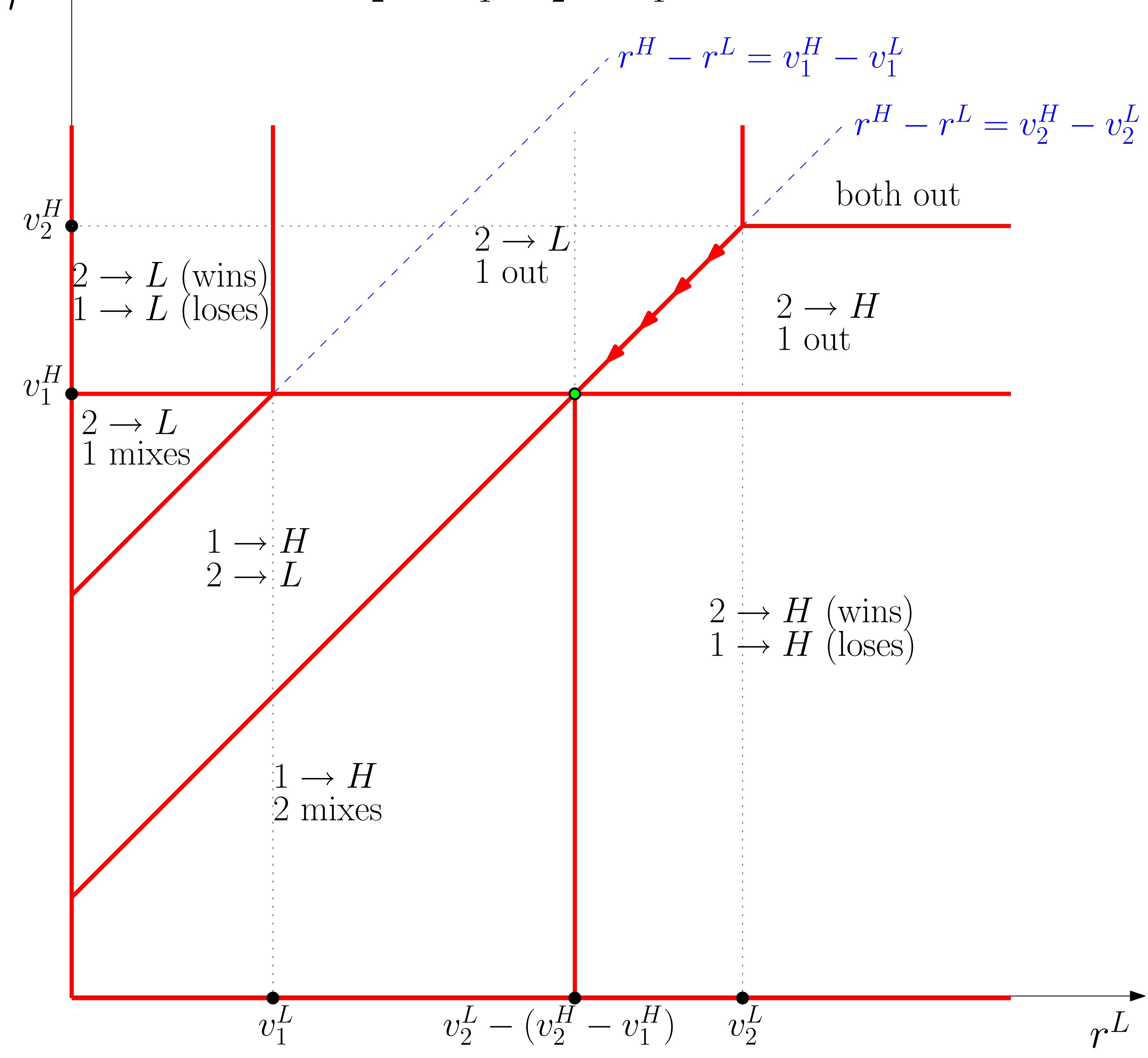


Figure 3.1: $v_2^H > v_1^H$, $v_2^L > v_1^L$

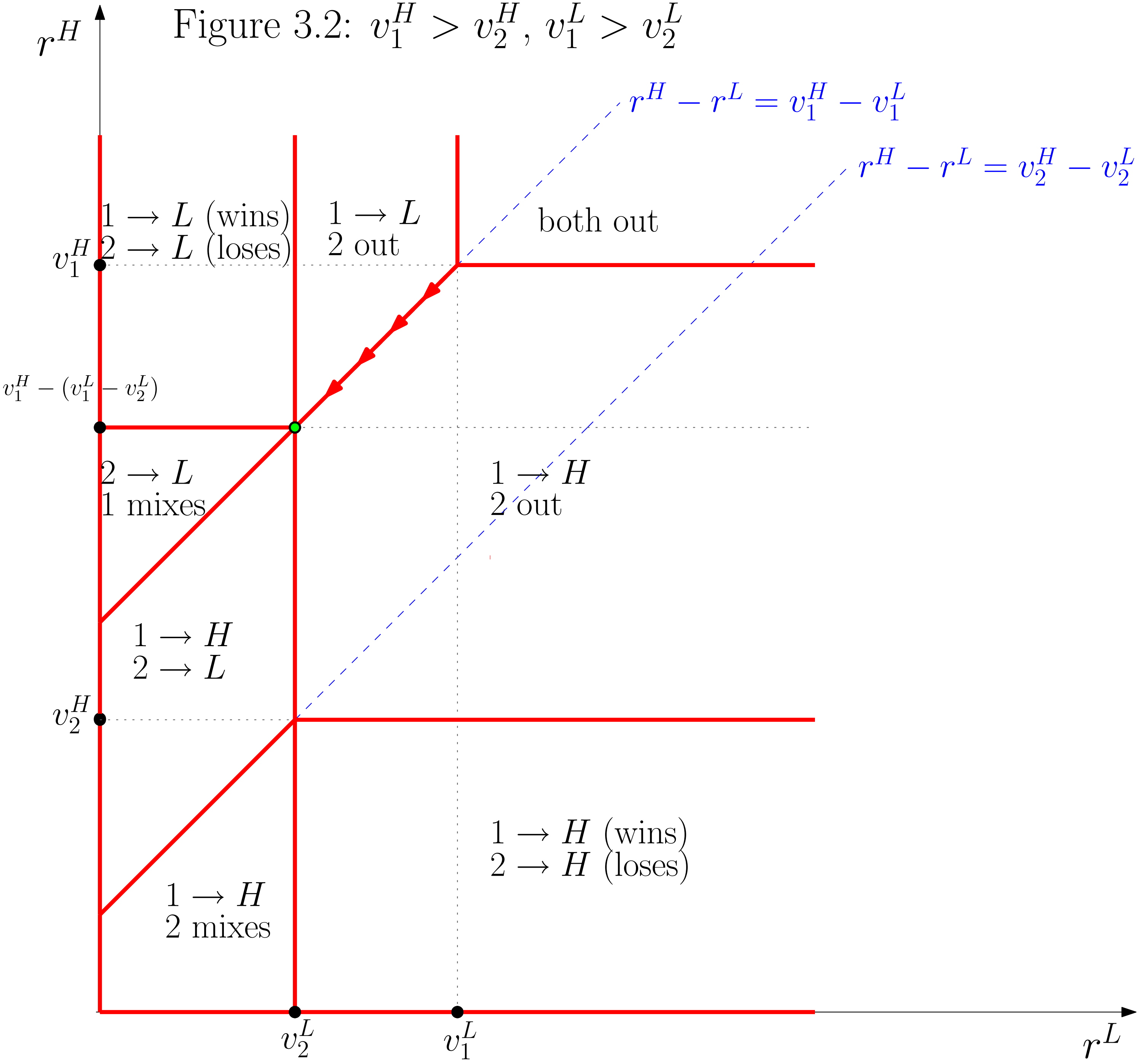


Figure 3.2: $v_1^H > v_2^H$, $v_1^L > v_2^L$

Figure 3.3: $v_1^H > v_2^H$, $v_2^L > v_1^L$