A Comparative Goodness-of-fit Analysis of Distributions of Some Lévy Processes and Heston Model to Stock Index Returns

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Abstract

In this paper, we investigate the goodness-of-fit of three Lévy processes, namely Variance-Gamma (VG), Normal-Inverse Gaussian (NIG) and Generalized Hyperbolic (GH) distributions, and probability distribution of the Heston model to index returns of twenty developed and emerging stock markets. Furthermore, we extend our analysis by applying a Markov regime switching model to identify normal and turbulent periods. Our findings indicate that the probability distribution of the Heston model performs well for emerging markets under full sample estimation and retains goodness of fit for high volatility periods, as it explicitly accounts for the volatility process. On the other hand, the distributions of the Lévy processes, especially the VG and NIG distributions, generally improves upon the fit of the Heston model, particularly for developed markets and low volatility periods. Furthermore, some distributions yield to significantly large test statistics for some countries, even though they fit well to other markets, which suggest that properties of the stock markets are crucial in identifying the best distribution representing empirical returns.

Keywords: Variance-gamma model, Normal-inverse gaussian model, Generalized hyperbolic model, Heston model, Markov regime-switching model, Emerging markets

1 Introduction

Empirical performance of the Black-Scholes model has been subject to biases due to violations of its two major assumptions on the underlying stock price process: (i) stock prices follow a continuous path

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in time and their distribution is log-normal (ii) the volatility of the price process is constant over time. However, asset returns are leptokurtic and display volatility clustering[1].

These assumptions are relaxed to allow for discontinuities in stock price paths in the form of pure jump processes[2] and allowing for stochastic volatility to account for volatility smile or skewness in the literature.[3] Furthermore, multivariate Lévy processes and stochastic volatility models can incorporate the correlation between the stock prices and volatility processes, enabling them to capture possible leverage effects. [4]

In this paper, we compare the univariate goodness-of-fit of various return distributions which are assumed by widely used option pricing models, namely the Variance-Gamma distribution introduced by Madan and Senata [1990], the Normal-Inverse Gaussian distribution introduced by Barndorff-Nielsen [1995], the Generalized Hyperbolic distribution defined by Barndorff-Nielsen [1977][5] and the distribution of the stochastic volatility model developed by Heston [1993].[6] We conduct our analysis for twenty developed and emerging stock market indices. Furthermore, we examine the performance of these models under high and low volatility regimes for each stock market identified using Markov-switching model.

These models are well-known and widely used in the financial industry for option pricing. However, to the best of our knowledge, there is no comparative goodness-of-fit analysis of the return distributions of these models for an extensive data set covering both developed and emerging markets. Furthermore, the application of the Markov regime switching model allows us to observe cross-sectional similarities or differences between the volatility regimes in different markets.

Madan and Seneta [1987] introduce the symmetric Variance Gamma (VG) process building on Praetz [1972]. Madan and Seneta [1990] extend the Black-Scholes model by applying the VG process to the option pricing framework. Madan et al. [1998] conclude that the VG model reduces the pricing bias that exists in the Black-Scholes model. Daal and Madan [2005] provide an empirical examination of the VG model comparing with the Merton [1976] jump-diffusion and Black and Scholes [1973] models for foreign

[1]Chernov et al. [1999] and Cont [2001]

[2]Madan and Seneta [1990] and Barndorff-Nielsen [1997]

[3]Hull and White [1987], Scott [1987] and Heston [1993]

[4]For multivariate Lévy processes, see Luciano and Semeraro [2010] ,Luciano and Semeraro [2010], Luciano et al. [2014]; for multifactor stochastic volatility models, see Fouque and Lorig [2001]

[5]For the applications of hyperbolic distributions to finance, see Eberlein and Keller [1995] and Eberlein and Prause [2002].

[6]We do not evaluate the implied return distributions of option pricing models, we rather conduct an analysis which focuses on the physical probability measure and goodness of fit to historical returns.

currency options. They document that the VG model outperforms other models used in their study.

Barndorff-Nielsen [1995] introduces the NIG model and compares its goodness-of-fit with hyperbolic distributions using data from Danish stock market. Barndorff-Nielsen [1997] extends NIG process with stochastic volatility. Rydberg [1997] examines the performance of the NIG model for returns of major German and Danish stocks. An investigation of the relative performance of the NIG and VG models is provided in Figueroa-Lopez et al. [2011], where they evaluate the goodness-of-fit using high-frequency transaction data from the US equity market. They show that both models improves upon the normal distribution.

The probability distribution of asset returns under the VG and NIG models are parametric special cases of the generalized hyperbolic (GH) distribution. Barndorff-Nielsen and Blaesild [1981] introduce GH distribution building on Barndorff-Nielsen [1977]. Eberlein and Keller [1995], Hurst and Platen [1997], Eberlein and Ozkan [2003] and Behr and Potter [2009] investigate the goodness-of-fit of the generalized hyperbolic distribution to stock market returns. In particular, Behr and Potter [2009] examine the goodness-of-fit of the generalized hyperbolic distribution, the generalized logF distribution, and finite mixtures of Gaussians, to the stock returns of S&P 500. They show that all distributions provide reasonable fits to monthly returns, however for daily returns, generalized hyperbolic family is superior in terms of absolute differences between the empirical and estimated distributions. VG, NIG and GH model parameters can be estimated using a variety of methods, such as, method of moments, maximum likelihood estimation, empirical characteristic function method and minimum Chi-square method.[7]

Madan and Seneta [1987], Barndorff-Nielsen [1995], Seneta [2004], Daal and Madan [2005], and Figueroa-Lopez et al. [2011] examine into the goodness-of-fit of the VG and NIG model versus the normal distribution. Madan and Seneta [1987] compare the VG model with normal, stable and the Press compound events model using the Chi-square goodness-of-fit statistics for 19 stocks from Sydney Stock Exchange. They show that for 12 stocks, VG model improves upon the fit of Press, normal and stable processes. Similarly, Seneta [2004] fits the VG model to S&P 500 index returns using method of moments estimation and show that VG model is superior to the normal distribution. An empirical analysis of goodness-of-fit of the VG model is given in Rathgeber et al. [2013] utilizing the Markov regime-switching model for Dow stocks.

[7] For details see Madan and Seneta [1987], Madan and Seneta [1990], Seneta [2004], and Finlay and Seneta [2008].

Dragulescu and Yakovenko [2002] derive a closed form expression for the probability distribution function of the Heston model and evaluates its goodness-of-fit to the Dow-Jones Industrial Average returns from January 04, 1982, to December 31, 2001. Silva and Yakovenko [2003] perform a similar analysis for NASDAQ and S&P 500 returns. Daniel et al. [2005] show that the Heston model does not significantly outperform the normal distribution with constant volatility for long time-horizons (40-250 days), yet provides a better fit for shorter time intervals (1-5 days).

This paper is organized as follows. In Section 2, models and their probability distribution functions for log-returns are briefly presented. In Section 3, we describe our dataset and implementation of the Markov regime-switching model. Section 4 presents the goodness-of-fit test results in developed and emerging markets. Section 5 concludes.

2 Stock price processes and the distributions of returns

2.1 Lévy Processes

Given the empirical facts for asset returns, Lévy processes have been widely used in modelling the dynamic behaviour of asset prices. Thus, we consider the following general representation of stock prices

$$S_t = S_{t-1} \exp(\Delta L_t), \tag{1}$$

where $\{L_t\}_{t\geq 0}$ is a generalized hyperbolic Lévy motion. For distributions with finite first moments, the process $dL_t = \sigma dW_t + dZ_t + \alpha dt$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion process and $\{Z_t\}_{t\geq 0}$ is a purely discontinuous martingale independent of $\{W_t\}_{t\geq 0}$. We assume that $L(t)$ is a Lévy process with marginals following the generalized hyperbolic distribution. Generalized hyperbolic distribution is a suitable choice for leptokurtic data, which contains some commonly used distributions in finance as parametric special cases, such as the Normal-inverse Gaussian (NIG) and the Variance-Gamma (VG) distributions. The generalized hyperbolic distribution has the density function given by

$$
\begin{aligned}
f_{GH}(x;\lambda,\mu,\alpha,\beta,\delta) &= \xi(\delta^2+(x-\mu)^2)^{(\lambda-0.5)/2}\exp\left(\beta(x-\mu)\right) \qquad (2)\\
&\times \; K_{\lambda-0.5}(\alpha\sqrt{\delta^2+(x-\mu)^2})\\
\text{where } \xi &= \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-0.5}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})}, \quad -\infty < x < \infty
\end{aligned}
$$

where $K_y(.)$ is the modified Bessel function of the third kind with index y and μ, β, δ, α, λ are the location, skewness, scale, tail, and the sub-family parameters, respectively. Two special cases of the generalized hyperbolic distribution, i.e. the Normal-inverse Gaussian and the Variance-Gamma distributions are widely used in financial applications [8], which are obtained with the parametric restrictions given by $\lambda = -0.5$ and $\delta = 0$, $\lambda > 0$, respectively.

The Lévy measure for the stock price process given in Equation 1 can be written explicitly as:

$$
l_{GH}(dz) = |z|^{-1}e^{\beta z}\left(\frac{1}{\pi^2}\int_0^{\infty}\frac{\exp(-\sqrt{2y+\alpha^2}|z|)}{J_{\lambda}^2(\delta\sqrt{2y})+Y_{\lambda}^2(\delta\sqrt{(2y)})}\frac{dy}{y}+\lambda e^{-\alpha|z|}\right)dz, \qquad (3)
$$

for $\lambda \geq 0$, and

$$
l_{GH}(dz) = |z|^{-1}e^{\beta z}\frac{1}{\pi^2}\int_0^{\infty}\frac{\exp(-\sqrt{2y+\alpha^2}|z|)}{J_{-\lambda^2}(\delta\sqrt{2y})+Y_{-\lambda^2}(\delta\sqrt{(2y)})}\frac{dy}{y}dz, \qquad (4)
$$

for $\lambda < 0$, where J_{λ} and Y_{λ} are the Bessel functions of the first and second kind, respectively with index λ.

There are various parametric special cases of the generalized hyperbolic distribution that yield to the return distributions widely used in financial modeling. For $\lambda = -0.5$ we obtain the normal-inverse Gaussian distribution, which was introduced to finance in Barndorff-Nielsen (1998) and has density

$$
f_{NIG}(x) = \frac{\alpha\delta K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\pi\sqrt{\delta^2+(x-\mu)^2}}e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \qquad (5)
$$

for $-\infty < x < \infty$ and $\alpha > |\beta| > 0$ with real model parameters μ (location), $\delta > 0$ (scale), β (skewness), and α (shape).

Another important special case is obtained for $\delta = 0$ and $\lambda > 0$, which gives us the Variance-Gamma

[8] See Madan et al. [1998] and Carr and Wu [2004]

distribution with the following density function:

$$f_{VG}(x) = \frac{2exp(\theta(x-c)/\sigma^2)}{\sigma\sqrt{2\pi}\nu^{1/\nu}\Gamma(1/\nu)}\left(\frac{|x-c|}{\sqrt{2\sigma^2/\nu+\theta^2}}\right)^{\frac{1}{\nu}-\frac{1}{2}} K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{|x-c|\sqrt{2\sigma^2/\nu+\theta^2}}{\sigma^2}\right), \tag{6}$$

for $-\infty < x < \infty$ with parameters c (location), θ (skewness), $\sigma > 0$ (scale), and $\nu > 0$ (shape). $K_\eta(.)$ is the modified Bessel function of the second kind.

For both the VG and NIG models, we use the maximum likelihood estimation method. The initial values for the maximum likelihood estimation are obtained by the method of moments estimator, which can be found in Seneta [2004] and Figueroa-Lopez et al. [2011].

Furthermore, since standard Brownian motion is a special case of the Lévy process, the stock price model of geometric Brownian motion given by

$$S_t = S_0 \exp(\alpha t + \sigma W_t), \tag{7}$$

can be obtained. Under the geometric Brownian motion model with the drift α and volatility $\sigma > 0$ of stock prices, log-returns are assumed to be normally distributed. We use this special case as the benchmark to verify the improvement achieved by assuming more general distributions for log-returns.

2.2 Heston stochastic volatility model

Heston [1993] model extends the Black-Scholes model by assuming a stochastic process for the volatility:

$$dS_t = S_t\mu dt + S_t\nu_t dW_t^1, \tag{8}$$

where μ is the long-term mean, σ_t is the volatility and W_t^1 is a standard Wiener process. It is assumed that the volatility follows a mean-reverting stochastic process given as:

$$d\nu_t = -\gamma(\nu_t - \theta)dt + \kappa\sqrt{\nu_t}dW_t^2. \tag{9}$$

where $\nu_t = \sigma_t^2$. In Equation (9), θ represents the long-term mean of ν_t, γ is the speed of mean reversion, and W_t^2 is a Wiener process. Two Wiener processes, W_t^1 and W_t^2 are not necessarily independent.

Allowing for possible correlation, we write

$$dW_t^2 = \rho dW_t^1 + \sqrt{1-\rho^2} dZ_t, \tag{10}$$

where Z_t is a Wiener process independent of W_t^1 and correlation coefficient is $\rho \in [-1, 1]$.

Heston [1993] model extends the Black-Scholes model by assuming a stochastic process for the volatility, which allows for correlation between the two noise process of the stock prices and the volatility. A negative correlation coefficient, ($\rho < 0$), is known as the leverage effect [9].

Under some restrictions on the stochastic process governing the volatility, one can obtain the marginal distribution of returns after integrating out the volatility term. Dragulescu and Yakovenko [2002] provides the solution for the stochastic differential equation as a Fourier integral and obtains the probability distribution of centered log-returns as:

$$P_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_x e^{ip_x + F_t(p_x)}. \tag{11}$$

The cumulative distribution function $F_t(p_x)$ is given as:

$$F_t(p_x) = \frac{\gamma \Gamma \theta t}{\kappa^2} - \frac{2\gamma\theta}{\kappa^2} \ln \left[\frac{\cosh \frac{\Omega t}{2} + \frac{\gamma}{\Omega} \sinh \frac{\Omega t}{2}}{\cosh \frac{\Omega t}{2} + \frac{\Gamma}{\Omega} \sinh \frac{\Omega t}{2}} \right] - \frac{2\gamma\theta}{\kappa^2} \ln \left[\cosh \frac{\Omega t}{2} + \frac{\Omega^2 - \Gamma^2 + 2\gamma\Gamma}{2\gamma\Omega} \sinh \frac{\Omega t}{2} \right],$$

where $\Gamma = \gamma + i\rho\kappa p_x$, ρ is the correlation coefficient of the two Wiener processes, $\Omega = \sqrt{\Gamma^2 + \kappa^2(p_x^2 - ip_x)}$ and γ, κ, θ, μ are the parameters of the Heston model.

Following Daniel et al. [2005], the parameters of the Heston model, $\gamma, \kappa, \theta, \mu$, are obtained by minimizing the distance between the theoretical and empirical mass functions via the following objective function

$$E = \sum_x |\log(P_t^*(x)) - \log(P_t(x))|^2, \tag{12}$$

where $P_t^*(x)$ denotes the empirical mass function and $P_t(x)$ is given in Equation (11). We did not trim the dataset in the fitting procedure as in Daniel et al. [2005]. In order to solve the optimization problem

[9] See Fouque et al. [2000]. For an economic interpretation of this phenomenon, reader is referred to Christie [1982].

Figure 1: Identification of high volatility and low volatility sub-periods for the US stock market index implementing a Markov regime switching model with weekly log-returns data

in Equation 12 we use the Nelder-Mead simplex search and the interior-reflective Newton methods.[10]

3 Data

Our data set consists of daily stock market index returns for several emerging and developed markets. These countries and their representative indices (in parenthesis) are: US (S&P 500), Japan (Nikkei), Germany (DAX), UK (FTSE 100), France (CAC 40), Australia (All Ordinaries), Singapore (Strait Times), Holland (AEX), Hong Kong (Hang Seng), Canada (S&P/TSX Composite), Turkey (Borsa Istanbul 100), China (Shanghai Stock Exchange Composite), Brazil (Ibovespa), India (BSE SENSEX), Russia (MICEX), Indonesia (Jakarta Stock Exchange Composite), Malaysia (FTSE Bursa Malaysia KLCI), Thailand (Stock Exchange of Thailand SET), South Africa (FTSE/JSE Africa All Share), and Mexico (Mexican Bolsa IPC) [11]. We calculate daily log-returns as $X_t := \ln(S_t/S_{t-1})$, where S_t is the closing value of each stock price index. Our dataset covers the period January 1998 to July 2013[12] with the descriptive statistics given in Table 1.

There can be significant volatility differences in stock markets during turbulent and normal periods, which is documented in Hamilton [1989] and Jeanne and Masson [2000]. Therefore, identification of different volatility regimes might be critical in the goodness-of-fit analysis. We use a regime switching model following Rathgeber et al. [2013] to identify high and low volatility regimes in each stock market.[13]. Our objective is to identify the periods when the stock indices are in low or high volatility states. For the calibration of the regime switching model, we use weekly log-returns since results tend to be more stable compared to the case of daily log-returns[14].

In Figure 1, we plot the smoothed probabilities for normal and turbulent regimes for the US index (S&P 500). For the US market, the whole sample is divided into four sub-samples with two normal periods (July 19, 2003 to August 3, 2007 and December 3, 2011 to July 15, 2013), and two turbulent

[10] See Lagarias et al. [1998], Coleman et al. [1994], and Coleman et al. [1996] for details.
[11] The dataset is obtained from Bloomberg.
[12] Time periods are selected based on the longest available historical data for each stock market.
[13] For a thorough discussion of regime switching models, see Hamilton [1989] and Hamilton [2005].
[14] Similar approach is used in Rathgeber et al. [2013], which only considers the VG model for US stocks.

Table 1: Descriptive statistics of daily log-returns for the developed and emerging market indices

Developed markets						
Market Index	Mean	Stdev.	Skewness	Kurtosis	Min.	Max.
US	0.0001	0.0129	-0.19	10.47	-0.095	0.11
Japan	0.0000	0.0153	-0.36	9.39	-0.12	0.13
Germany	0.0001	0.0159	-0.06	6.96	-0.09	0.11
UK	0.0000	0.0126	-0.13	8.48	-0.09	0.09
France	0.0000	0.0153	0.00	7.41	-0.09	0.10
Australia	0.0000	0.0099	-0.56	8.94	-0.09	0.05
Singapore	0.0000	0.0138	0.03	9.95	-0.09	0.13
Holland	0.0000	0.0152	-0.11	8.59	-0.09	0.10
Hong Kong	0.0002	0.0167	0.13	10.38	-0.13	0.13
Canada	0.0002	0.0118	-0.65	11.47	-0.10	0.09
Emerging markets						
Turkey	0.0000	0.0259	0.09	8.9467	-0.20	0.17
China	0.0000	0.0162	-0.08	7.17	-0.09	0.09
Brazil	0.0000	0.0213	0.53	16.88	-0.17	0.28
India	0.0000	0.0167	-0.13	8.56	-0.11	0.16
Russia	0.0000	0.0284	0.23	16.01	-0.20	0.27
Indonesia	0.0000	0.0168	-0.18	10.23	-0.13	0.13
Malaysia	0.0000	0.0080	-1.10	16.77	-0.10	0.04
Thailand	0.0000	0.0165	-0.01	10.80	-0.16	0.11
South Africa	0.0000	0.0131	-0.27	6.49	-0.08	0.07
Mexico	0.0000	0.0152	0.14	8.17	-0.10	0.12

Figure 2: Empirical versus fitted distributions of the Heston, VG, NIG and GH models for the US index (S& P 500)

periods (January 2, 1998 to July 18, 2003 and August 4, 2007 to December 2, 2011). The plots of the Markov regime switching model for the rest of the stock markets are similar to Figure 1 and thus not presented here.[15]

4 Goodness-of-fit results

We measure the distance between the empirical asset returns and the theoretical distributions under different models employing the Chi-square (CS) and Anderson-Darling (AD)[16] goodness-of-fit tests. CS test accounts for the number of model parameters and thus allows for the estimation using the same empirical dataset being tested. Although this is not true for the AD test, it provides a good measure of distance between the empirical and theoretical distributions by putting more weight to the tails of the distribution. Both for the CS and AD statistics, a smaller value suggests a better fit to the empirical distribution. Some studies utilize Kolmogorov-Smirnov test statistic, however, this test statistic is based on the maximal distance between the empirical and theoretical distributions focusing on a single maximal deviation point, rather than testing the overall fit. Therefore, instead of using the Kolmogorov-Smirnov test statistic, we utilize the CS and AD test statistics that reflects the overall fit of the distributions.

For the CS test, we consider the set of bins given in Madan and Seneta [1987], i.e. $(-\infty, -1.0, -0.75, -0.25, -0.1, 0.1, 0.25, 0.75, 1.00, \infty)$, however, our numerical results are robust to the selection of alternative bins in CS test. The degrees of freedom of the CS test statistic equals to number of bins $- m - 1$, where m is the number of parameters of the model. For the estimation of all models, we utilize the mean-subtracted data and the sample mean is used to pin down the location parameters.

We observe that for the S&P 500 Index, normality is rejected by the CS test for the full sample and all the sub-periods. However, the sub-period December 3, 2011 to July 15, 2013, i.e. the aftermath of the financial crisis in the US, the CS value for the normal distribution is reduced to 17.86, whereas

[15] The plots for other countries are available upon request.
[16] See Anderson and Darling [1952]

Table 2: US: Estimated parameters and goodness-of-fit statistics of the normal, VG, NIG distributions and distribution for the Heston model. Chi-square (CS) and Anderson-Darling (AD) goodness-of-fit measures are calculated. "*" means rejected by the CS test at 95% confidence level.

Period	**Estimated Parameters**					**Goodness-of-fit Statistics**	
	Normal Distribution						
Period	Mean	Std				CS	AD
1998/1-2013/7	0.0001	0.0132				407.6364*	55.5000
1998/1-2003/7	0.0000	0.0136				26.5123*	3.2415
2003/7-2007/8	0.0004	0.0070				40.4733*	3.4758
2007/8-2011/12	-0.0002	0.0177				172.1016*	21.208
2011/12-2013/7	0.0008	0.0080				17.8573*	1.6550
	VG model						
Period	c	θ	σ	ν		CS	AD
1998/1-2013/7	0.0011	-0.0010	0.0127	1.0290		4.6626	0.6594
1998/1-2003/7	-0.0001	0.0001	0.0136	0.3979		4.1797	0.2957
2003/7-2007/8	0.0016	-0.0012	0.0070	0.5464		11.0665	0.4082
2007/8-2011/12	0.0012	-0.0014	0.0171	1.2653		8.5837	0.4511
2011/12-2013/7	0.0008	0.0000	0.0081	0.6120		3.3082	0.1711
	NIG model						
Period	μ	β	δ	α		CS	AD
1998/1-2013/7	0.0005	-2.0660	0.0093	54.2137		15.1960*	1.4347
1998/1-2003/7	-0.0002	1.0458	0.0192	103.8221		3.1204	0.2783
2003/7-2007/8	0.0022	-36.9600	0.0102	216.6714		10.0533	0.7496
2007/8-2011/12	0.0005	-2.0564	0.0106	32.9527		21.6620*	1.3616
2011/12-2013/7	0.0015	-10.4134	0.0097	149.2056		7.6093	0.2271
	Heston model						
Period	θ	γ	κ	ρ		CS	AD
1998/1-2013/7	0.0002	34.8013	0.7780	-0.0022		21.8884*	4.0343
1998/1-2003/7	0.0002	81.5843	0.7111	0.0644		4.6479	0.6061
2003/7-2007/8	0.0001	56.1002	0.7503	-0.0542		19.9991*	1.9254
2007/8-2011/12	0.0005	15.4621	0.7778	-0.0015		20.4298*	3.6003
2011/12-2013/7	0.0135	0.2382	0.7908	0.0592		30.1151*	5.0317
	GH model						
Period	location	scale	tail index	shape	skewness	CS	AD
1998/1-2013/7	0.0002	0.0129	-0.0002	0.3838	-0.0683	19.59*	1.22
1998/1-2003/7	0.0000	0.0134	0.0003	1.4879	-0.0006	10.71	0.47
2003/7-2007/8	0.0004	0.0069	0.0001	1.3084	-0.1074	28.96	0.94
2007/8-2011/12	-0.0002	0.0177	0.0002	0.2496	-0.1153	19.47	0.54
2011/12-2013/7	0.0008	0.0080	0.0001	0.8648	-0.0027	10.48	0.56

the critical value at 99% confidence level is 16.81. Even though normality assumption is rejected for each sub-period, there are significant differences in terms of the quality of the fit obtained. NIG model is rejected for the full sample, however, we fail to reject in all the sub-periods except during the high volatility and financial crisis period of August 4, 2007 to December 2, 2011. Heston model is rejected for the full sample consistent with the results of Daniel et al. [2005]. However, considering the sub-periods, Heston model cannot be rejected in three out of four sub-periods. Based on the CS test, we fail to reject the VG model for the full sample and for all sub-periods. Similar results are obtained using the VG model for the AD test statistics. Based on the CS and AD goodness-of-fit measures for the SPX index, VG model performs the best.

As an example we plot the fitted distributions for the S&P 500 index in Figure 2, where the VG distributions seems to fit relatively better compared to the alternatives considered, which can also be verified with the AD test statistic given in Table 2 for the whole sample size.

In the maximum likelihood estimation for the VG and NIG models, the initial values of the parameters are obtained from the method of moments estimation. However, for the Heston model and the GH distribution we need to provide the initial values based on an educated guess. Often, it is not an easy task to provide good initial values such that the optimization routine converges to a global optima, which makes the parameter estimation computationally more expensive and more sensitive to the choice of initial values.

4.1 Emerging versus developed markets

In Table 3, we compare standard deviations of sub-samples under different volatility regimes in emerging and developed countries. We observe that mean volatility is higher in emerging markets for both types of regimes. Cross-country differences are significantly larger within the group of emerging markets, where standard deviation of volatility is higher under both high and low volatility regimes. These differences suggest that performance of alternative models may vary both across countries and different volatility periods.

First, we evaluate the performance of alternative models for the full samples, without accounting for different volatility regimes. Table 4 provides Anderson Darling (AD) and Chi-square (CS) test statistics for the full set of countries. We observe that all four models outperforms normal distribution, consistent

Table 3: Volatility under High and Low Regimes : Emerging vs. Developed Countries

Developed			
Volatility	High	Low	Difference
N	19	13	
Min	0.0123	0.0067	
Max	0.0276	0.0128	
Mean	0.0167	0.0091	0.0076
Std	0.0036	0.0018	
Emerging			
Volatility	High	Low	Difference
N	16	15	
Min	0.0135	0.0081	
Max	0.0491	0.0192	
Mean	0.0260	0.0126	0.0134
Std	0.0109	0.0037	

with the previous literature for both tests. We provide the best performing distributions for each sample under the "Best" column, along with the test statistics. Our results suggests that the evaluation of the goodness of fit critically depends on the selection of the test method as the best performing distribution varies for CS and AD tests. The AD test is more sensitive compared to the CS test as it uses the specific distribution being tested to calculate the critical values and generally more powerful than the CS test. However, our main purpose is to assess the goodness of fit of alternative models, rather than comparing power of different tests, thus, we choose to provide both commonly used test statistics for the countries in our sample.

Our results for the best performing distribution vary significantly between developed and emerging markets. For the developed markets, distributions of Lévy processes outperform the probability distribution of the Heston model, as the empirical returns of four countries and two countries can be best characterized by the Heston model, respectively, depending on the test-statistic. However, for the emerging markets, Heston model is selected more frequently, for five countries according to the CS test and for four countries using the AD test. As we have presented in Table 3, emerging markets exhibit a higher volatility of returns, therefore the probability distribution of the Heston model provides a better fit since it accounts explicitly for the volatility in the return process and we are not differentiating between high and low volatility regimes.

Among the class of distributions of Lévy processes, for the developed markets, the VG distribution is selected as the best for five countries using the CS test and for three countries based on the AD test. However, the VG distribution is dominated by the other Lévy processes and the probability distribution of the Heston model for the emerging markets. However, according to the CS test, NIG, and based on AD test NIG and GH, improve upon the fit of VG, we observe that VG still performs well with test statistics close to the "Best".

In Table 5, we present the descriptive statistics of the CS and AD test statistics obtained for emerging and developed markets, which is also categorized with respect to the high and low volatility sub-periods. For each model, we provide the number of times that the given model is the best performer as "Rank1" statistic. Furthermore, Table 6 provides a summary of best performing distributions with respect to different test statistics and volatility regimes.

According to the AD test, for developed markets, in both high and low volatility regimes, comparing the mean values of test statistics, distributions of Lévy processes yield to relatively close values and outperform the probability distribution of the Heston model, however we do not observe such a pattern for the CS test. For the case of emerging markets, again based on the comparison of the mean values of AD test statistic, VG and NIG distributions provide better fits than alternative distributions in both high and low volatility periods. However, during high volatility periods, Heston model is superior.

According to the "Rank" statistics, the probability distribution of the Heston model performs relatively better for the emerging markets, and for the full-sample and high-volatility periods, consistent with the ability of the distribution to account for the volatility of index returns. On the other hand, considering the mean values of test statistics, the VG distribution is the best performer in most cases and for the remaining cases, it yields close to the best result.

We observe that identification of high and low volatility regimes and application of the models for these sub-periods, provides a significant improvement in the goodness-of fit statistics. Our results suggest that empirical index returns fits better to the VG distribution for all sub-samples in developed markets and for low volatility periods in emerging markets. For the high volatility regimes in emerging markets and for the full sample, the distribution of the Heston model improves upon the fit of the VG distribution. One potential explanation for the relatively poor performance of the VG model in high volatility regimes is that the VG model does not provide a sufficient fit whenever the leptokurtosis is more pronounced.

Moreover, there are significant volatility differences between developed and emerging markets, which hinders the fit of the VG model, the probability distribution of the Heston model performs relatively better in high volatility regimes due to pronounced leverage effects.

Finally, log-likelihood ratio test is applied [17] in order to test the nested parametric restrictions, i.e. VG and NIG, of the GH distribution. Results given in Table 7 shows that in general GH distribution in many cases is not giving significant improvements over the VG and NIG distributions. For the developed markets in 6 out of 10 countries we fail to reject the VG distribution, whereas in the emerging markets we fail to reject the NIG distribution in 7 out of 10 countries. Since either VG or NIG model seems to be a good substitute for the GH distribution, the use a more complex distribution is not appealing. Furthermore, there are not readily available formulas for pricing options under the GH distributed Lévy processes as well.

5 Conclusion

In this paper, we compare the univariate goodness-of-fit of the return distributions which are assumed by widely used option pricing models namely the Variance-Gamma distribution, the Normal-Inverse Gaussian distribution, the Generalized Hyperbolic distribution, and the distribution of the Heston [1993] stochastic volatility model to empirical returns using a dataset from twenty emerging and developed stock market indices. We extend our analysis with an application of the Markov regime switching model to evaluate the performance of alternative models under different volatility regimes.

First, we conduct a full sample estimation prior to the identification of high and low volatility periods and implement the CS and AD tests. We observe that, probability distributions of the Lévy processes improves upon the fit of the probability distribution of the Heston model for developed market stock indices. However, for emerging markets, Heston model performs relatively better as it explicitly accounts for the volatility in the return process. Among the class of distributions of Lévy processes, VG distribution performs relatively better, but, the VG distribution is dominated by the other Lévy processes and the probability distribution of the Heston model for the emerging markets, even though the VG distribution still provides test statistics close to the "Best".

[17] Likelihood ratio test for the VG and NIG distributions as special cases of the GH distribution is applied in R using the ghyp package.

Second, we reapply our analysis after the identification of different volatility regimes. For both high and low volatility regimes, we observe that distributions of Lévy processes yield to relatively close AD test statistics and improves upon the fit of the probability distribution of the Heston model. A cross-country comparison suggests that the CS test does not support this observation. For emerging market stock indices, AD test statistics for VG and NIG distributions provide better fits than alternative distributions in both high and low volatility periods.

Third, although the GH distribution has more number of parameters, in most of the countries either NIG or VG distribution is providing a sufficiently good fit to the log-returns in comparison to the GH distribution. Furthermore, due to the analytical tractability of Lévy processes in option pricing with NIG or VG distributed marginals, the use of GH distribution is not very appealing. We should also note that the maximum likelihood estimation for the VG and NIG distributions, with the input of good initial values from method of moments estimation, is more efficient and robust compared to the maximum likelihood estimation of the GH distribution, where there is no readily available formula for the method of moments estimation. Therefore, we conclude that for modeling the daily index returns parametric special cases of the GH distribution, i.e. the VG or NIG distributions, can be used.

Our analysis suggest that achieving the best fit to the empirical returns critically depends on the selection of the probability distribution and the test statistic used to evaluate the goodness-of-fit. We observe that, even though the CS and AD test statistics agree for the majority of the countries, there can be significant discrepancies in their values for different countries. For volatile markets, independent of the test statistic used, if different volatility regimes are not controlled for, the Heston model may be a suitable choice. However, once we identify high and low volatility periods, the success of the probability distribution of the Heston model diminishes for low volatility periods. Furthermore, distributions of the Lévy processes, especially the VG and NIG distributions, generally outperform the probability distribution of the Heston model except some sub-domains of high volatility periods and the full sample, as presented in Table 6. We also observe that some distributions yield to significantly large test statistics for some countries, even though they fit well to other markets, which suggest that properties of the stock markets are crucial in identifying the best distribution representing the empirical returns.

Table 4: Chi-square and Anderson-Darling measures of goodness-of-fits for full samples of developed and emerging markets.

Panel A. Developed Markets												
	Chi-Square						Anderson-Darling					
Country	VG	NIG	Heston	GH	Normal	Best	VG	NIG	Heston	GH	Normal	Best
US	4.66	15.20	21.89	49.59	407.64	VG	0.66	1.43	4.03	1.22	∞	VG
Japan	7.47	9.34	15.00	145.18	168.60	VG	0.99	0.80	2.77	3.80	∞	NIG
Germany	2.49	19.73	23.57	43.12	248.15	VG	0.40	2.56	5.07	1.10	36.73	VG
UK	14.11	9.80	4.47	21.40	278.82	Heston	0.71	0.72	0.16	0.62	39.74	Heston
France	5.67	7.06	10.25	12.38	234.33	VG	0.66	1.61	2.42	0.54	34.59	GH
Australia	11.64	8.51	4.58	5.91	224.66	Heston	0.79	0.35	0.73	0.37	36.67	NIG
Singapore	15.53	10.28	7.56	5.13	350.03	GH	1.09	0.91	0.29	0.58	∞	Heston
Netherlands	13.79	13.79	11.73	15.94	342.76	Heston	1.36	1.28	2.49	0.56	54.49	GH
Hong Kong	10.32	27.70	9.47	121.74	352.32	Heston	0.49	1.96	0.90	3.32	47.15	VG
Canada	25.64	14.64	196.98	22.14	348.57	NIG	2.05	1.57	61.93	1.40	56.90	GH
Panel B. Emerging Markets												
	Chi-Square						Anderson-Darling					
Country	VG	NIG	Heston	GH	Normal	Best	VG	NIG	Heston	GH	Normal	Best
Turkey	10.26	4.76	8.08	43.94	267.52	NIG	1.25	0.50	0.39	1.10	49.33	Heston
China	6.68	9.54	6.00	62.75	293.14	Heston	0.41	0.54	1.09	4.12	39.66	VG
Brazil	10.72	11.87	8.31	39.23	147.80	Heston	1.39	1.39	0.61	2.03	∞	Heston
India	4.50	5.53	1.53	109.32	222.42	Heston	0.61	1.12	0.20	8.06	∞	Heston
Russia	15.10	14.37	13.29	99.92	471.08	Heston	2.77	4.55	4.87	2.70	∞	GH
Indonesia	19.15	7.59	10.84	87.84	368.55	NIG	1.89	0.55	2.11	1.99	58.55	NIG
Malaysia	71.43	13.17	160.83	58.76	1170.60	NIG	4.30	0.81	41.54	2.19	∞	NIG
Thailand	8.16	3.66	8.99	50.35	261.20	NIG	0.91	0.43	0.37	2.92	41.75	Heston
South Africa	13.74	5.56	9.51	13.96	146.69	NIG	1.05	0.42	2.55	1.04	24.59	NIG
Mexico	14.62	19.90	13.10	20.77	293.09	Heston	0.97	2.60	1.48	0.90	44.27	GH

Table 5: Summary of goodness-of-fit statistics obtained for emerging and developed markets

Panel A. Summary Statistics for Chi-Square Tests

Developed Markets	Full Sample (N=10)					High Volatility (N=19)					Low Volatility (N=13)				
	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std
Heston	4	4.466	196.977	30.549	58.838	4	1.498	32.166	8.621	7.568	4	0.878	26.384	8.760	7.387
NIG	1	7.057	27.696	13.603	6.256	4	2.336	21.662	8.096	5.363	2	2.388	27.651	9.382	7.871
VG	4	2.490	25.639	11.132	6.720	5	2.316	22.475	6.381	4.511	5	1.672	12.624	6.189	4.062
GH	1	5.130	145.180	44.253	49.497	6	2.129	66.483	21.031	20.883	2	1.926	48.439	14.656	14.291

Emerging Markets	Full Sample (N=10)					High Volatility (N=16)					Low Volatility (N=15)				
	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std
Heston	5	1.530	160.832	24.048	48.182	6	1.558	19.037	8.147	5.318	4	1.920	83.664	12.225	20.093
NIG	5	3.659	19.901	9.596	5.190	7	2.705	27.570	8.652	6.552	6	3.650	17.266	7.826	4.395
VG	0	4.501	71.426	17.435	19.467	3	1.567	24.572	9.110	6.826	4	1.818	15.390	7.022	3.783
GH	0	13.960	109.320	58.684	32.031	0	7.648	123.128	46.911	39.108	1	2.812	109.081	40.259	36.137

Panel B. Summary Statistics for Anderson-Darling Tests

Developed Markets	Full Sample (N=10)					High Volatility (N=19)					Low Volatility (N=13)				
	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std
Heston	2	0.155	61.933	8.079	18.991	4	0.200	15.549	1.710	3.433	0	0.248	18.223	3.038	5.435
NIG	2	0.350	2.555	1.319	0.653	8	0.175	1.826	0.612	0.482	2	0.172	1.303	0.458	0.347
VG	3	0.400	2.051	0.919	0.489	2	0.156	1.927	0.593	0.430	9	0.148	0.648	0.291	0.157
GH	3	0.373	3.798	1.351	1.217	5	0.233	1.652	0.570	0.359	2	0.173	1.672	0.704	0.594

Emerging Markets	Full Sample (N=10)					High Volatility (N=16)					Low Volatility (N=15)				
	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std	Rank1	Min	Max	Mean	Std
Heston	4	0.195	41.544	5.520	12.736	1	0.346	2.721	1.146	0.798	2	0.130	25.742	2.287	6.500
NIG	3	0.415	4.554	1.292	1.328	5	0.145	4.510	1.019	1.156	6	0.117	0.755	0.358	0.212
VG	1	0.410	4.296	1.554	1.176	4	0.147	1.858	0.648	0.508	6	0.137	0.647	0.306	0.160
GH	2	0.898	8.063	2.706	2.124	6	0.336	18.220	2.213	4.536	1	0.074	6.280	1.371	1.553

Table 6: Performance Summary of Different Models

Developed	Statistic	Full	High	Low
Chi-Square	Mean	VG	VG	VG
	Min	VG	Heston	Heston
	Max	VG	VG	VG
	Rank1	Heston	Heston/NIG	VG
A-D	Mean	VG	VG	VG
	Min	Heston	VG	VG
	Max	VG	VG	VG
	Rank1	VG	NIG	VG
Emerging	**Statistic**	**Full**	**High**	**Low**
Chi-Square	Mean	NIG	Heston	VG
	Min	Heston	Heston	VG
	Max	NIG	Heston	VG
	Rank1	Heston/NIG	NIG	NIG
A-D	Mean	NIG	VG	VG
	Min	Heston	NIG	NIG
	Max	VG	VG	VG
	Rank1	Heston	GH	VG/NIG

Table 7: P-values for the likelihood ratio tests for the whole samples of log-returns in each country, where GH is alternative model.

Developed Markets			Emerging Markets		
	VG	NIG		VG	NIG
US	1.0000	0.0000**	Turkey	0.0000**	0.1213
Japan	1.0000	0.0000**	China	0.0096**	0.1110
Germany	0.3964	0.0000**	Brazil	0.0000**	0.0000**
UK	1.0000	0.2574	India	0.0000**	0.6002
France	0.0146*	0.6824	Russia	0.0000**	0.0013**
Australia	0.0003**	0.3254	Indonesia	0.0000**	0.2079
Singapore	1.0000	0.0000**	Malaysia	0.0000**	0.6706
Holland	0.0000**	0.9410	Thailand	0.0000**	1.0000
Hong Kong	1.0000	0.0000**	South Africa	0.0000**	0.0181*
Canada	0.0000**	0.0436*	Mexico	0.0000**	0.9183

“*” Rejected at 5% significance level, “**” rejected at 1% significance level.

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